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ON THE MAIN EQUATION OF INVERSE STURM-LIOUVILLE OPERATOR WITH DISCONTINUOUS COEFFICIENT

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ABSTRACT. In this paper, the main equation which has an important role in solution of inverse problem for boundary value problem is obtained and according to spectral data, the uniqueness of solution of inverse problem is proved.
Key words and phrases. inverse Sturm-Liouville Operator; discontinuous coefficient.

1. INTRODUCTION

In many practices, spectral problems are faced for differential equations which have discontinuous coefficient and discontinuity conditions in interval ([1]-[8]). These problems generally emerge in physics, mechanics and geophysics in non-homogeneous and discontinuous environments.

We consider a heat problem in a rod which is composed of materials having different densities. In the initial time, let the temperature be given arbitrary. Let the temperature be zero in one end of the rod and the heat be isolated at the other end of the rod. In this case the heat flow in non-homogeneous rod is expressed with the following boundary problem:

$$\rho(x)\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+q(x)\,u, \qquad 0<x<\pi,\ t>0,$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0, \qquad u|_{x=\pi}=0, \quad t>0,$$

where $\rho(x)$, $q(x)$ are physical parameters and have specific properties. For instance, $\rho(x)$ defines the density of the material and piecewise-continuous function. Applying the method of separation of variables to this problem, we get the spectral problem below:

$$-y''+q(x)y=\lambda^2\rho(x)y,\ 0\le x\le\pi \tag{1}$$

(2) $$y'(0) = y(\pi) = 0,$$

here $q(x) \in L_2(0,\pi)$ is a real-valued function, $\rho(x)$ piecewise-continuous function the following:

(3) $$\rho(x) = \begin{cases} 1, & 0 \le x \le a, \\ \alpha^2, & a < x \le \pi \end{cases}$$

λ is spectral parameter and $a(1+\alpha) > \pi\alpha$.

When $\rho(x) \equiv 1$ or $\alpha = 1$, that is, in continuous case, the solution of inverse problem is given in [9]-[19]. The spectral properties of Sturm-Liouville opeator with discontinuos coefficient in different boundary conditions are examined in [20]-[23].

In this study, the main equation is obtained which has an important role in solution of inverse problem for boundary value problem and according to spectral data, the uniqueness of solution of inverse problem is proved. Similar problems are examined for the equation (1) with different boundary conditions in [24].

It was proved (see [25]), that the solution $\varphi(x,\lambda)$ of the equation (1) with initial conditions $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = 0$ can be represented as

(4) $$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^{\mu^+(x)} A(x,t)\cos\lambda t dt,$$

where $A(x,t)$ belongs to the space $L_{2,\rho}(0,\pi)$ for each fixed $x \in [0,\pi]$ and is related with the coefficient $q(x)$ of the equation (1) by the formula:

(5) $$\frac{d}{dx}A(x,\mu^+(x)) = \frac{1}{4\sqrt{\rho(x)}}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right)q(x),$$

(6) $$\varphi_0(x,\lambda) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right)\cos\lambda\mu^+(x) + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right)\cos\lambda\mu^-(x)$$

is the solution of (1) when $q(x) \equiv 0$,

(7) $$\mu^+(x) = \pm x\sqrt{\rho(x)} + a\left(1 \mp \sqrt{\rho(x)}\right)$$

It is similary shown in [24], [21] that the roots of the equation $\varphi(\pi,\lambda) = 0$ have the following form

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n},\ \lambda_n \ge 0,$$

where $\{\lambda_n^0\}^2$ are the eigenvalues of problem (1), (2) when $q(x) \equiv 0$, d_n is a bounded sequence, $k_n \in l_2$ and norming constants:

$$\alpha_n = \int_0^\pi \rho(x)\varphi^2(x,\lambda_n)dx.$$

2. Main Equation

Theorem 1. *For each fixed $x \in [0,\pi]$ the kernel $A(x,t)$ from the representation (4) satisfies the following linear functional integral equation*

$$\frac{2}{1+\sqrt{\rho(t)}}A\left(x,\mu^+(t)\right) + \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A\left(x,2a-t\right)+$$

$$+F(x,t) + \int_0^{\mu^+(x)} A(x,\xi)F_0(\xi,t)d\xi = 0, \quad 0<t<x \tag{8}$$

where

$$F_0(x,t) = \sum_{n=1}^{\infty}\left(\frac{\varphi_0(t,\lambda_n)\cos\lambda_n x}{\alpha_n} - \frac{\varphi_0(t,\lambda_n^0)\cos\lambda_n^0 x}{\alpha_n^0}\right) \tag{9}$$

$$F(x,t) = \frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)F_0(\mu^+(x),t) + \frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)F_0(\mu^-(x),t) \tag{10}$$

$\{\lambda_n^0\}^2$ are eigenvalues and α_n^0 are norming constants of the boundary value problem (1), (2) when $q(x) \equiv 0$.

Proof. From (4) we have

$$\varphi_0(x,\lambda) = \varphi(x,\lambda) - \int_0^{\mu^+(x)} A(x,t)\cos\lambda t dt. \tag{11}$$

It follows from (4) and (11) that

$$\sum_{n=1}^{N}\frac{\varphi(x,\lambda_n)\varphi_0(t,\lambda_n)}{\alpha_n} = \sum_{n=1}^{N}\left(\frac{\varphi_0(x,\lambda_n)\varphi_0(t,\lambda_n)}{\alpha_n}+\right.$$

$$\left.+\frac{\varphi_0(t,\lambda_n)}{\alpha_n}\int_0^{\mu^+(x)} A(x,\xi)\cos\lambda_n\xi d\xi\right) =$$

$$= \sum_{n=1}^{N}\left(\frac{\varphi_0(x,\lambda_n)\varphi_0(t,\lambda_n)}{\alpha_n} - \frac{\varphi_0(x,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}\right)+$$

$$+\sum_{n=1}^{N}\frac{\varphi_0(x,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}+$$

$$+\int_0^{\mu^+(x)} A(x,\xi)\sum_{n=1}^{N}\left(\frac{\varphi_0(t,\lambda_n)\cos\lambda_n\xi}{\alpha_n} - \frac{\varphi_0(t,\lambda_n^0)\cos\lambda_n^0\xi}{\alpha_n^0}\right)d\xi+$$

$$+\int_0^{\mu^+(x)} A(x,\xi)\sum_{n=1}^{N}\frac{\varphi_0(t,\lambda_n^0)\cos\lambda_n^0\xi}{\alpha_n^0}d\xi$$

$$\sum_{n=1}^{N}\frac{\varphi(x,\lambda_n)\varphi_0(t,\lambda_n)}{\alpha_n}=\sum_{n=1}^{N}\frac{\varphi(x,\lambda_n)\varphi(t,\lambda_n)}{\alpha_n}-$$

$$-\int_0^{\mu^+(t)} A(t,\xi)\sum_{n=1}^{N}\frac{\varphi(x,\lambda_n)\cos\lambda_n\xi}{\alpha_n}d\xi.$$

Using the last two equalities, we obtain

$$\sum_{n=1}^{N}\left(\frac{\varphi(x,\lambda_n)\varphi(t,\lambda_n)}{\alpha_n}-\frac{\varphi_0(x,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}\right)=$$

$$=\sum_{n=1}^{N}\left(\frac{\varphi_0(x,\lambda_n)\varphi_0(t,\lambda_n)}{\alpha_n}-\frac{\varphi_0(x,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}\right)+$$

$$+\int_0^{\mu^+(x)} A(x,\xi)\sum_{n=1}^{N}\frac{\varphi_0(t,\lambda_n^0)\cos\lambda_n^0\xi}{\alpha_n^0}d\xi+$$

$$+\int_0^{\mu^+(x)} A(x,\xi)\sum_{n=1}^{N}\left(\frac{\varphi_0(t,\lambda_n)\cos\lambda_n\xi}{\alpha_n}-\frac{\varphi_0(t,\lambda_n^0)\cos\lambda_n^0\xi}{\alpha_n^0}\right)d\xi+$$

$$+\int_0^{\mu^+(t)} A(t,\xi)\sum_{n=1}^{N}\frac{\varphi(x,\lambda_n)\cos\lambda_n\xi}{\alpha_n}d\xi,$$

or

(12) $$\Phi_N(x,t)=I_{N1}(x,t)+I_{N2}(x,t)+I_{N3}(x,t)+I_{N4}(x,t),$$

where

$$\Phi_N(x,t):=\sum_{n=1}^{N}\left(\frac{\varphi(x,\lambda_n)\varphi(t,\lambda_n)}{\alpha_n}-\frac{\varphi_0(x,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}\right),$$

$$I_{N1}(x,t):=\sum_{n=1}^{N}\left(\frac{\varphi_0(x,\lambda_n)\varphi_0(t,\lambda_n)}{\alpha_n}-\frac{\varphi_0(x,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}\right),$$

$$I_{N2}(x,t):=\int_0^{\mu^+(x)} A(x,\xi)\sum_{n=1}^{N}\frac{\varphi_0(t,\lambda_n^0)\cos\lambda_n^0\xi}{\alpha_n^0}d\xi,$$

$$I_{N3}(x,t):=\int_0^{\mu^+(x)} A(x,\xi)\sum_{n=1}^{N}\left(\frac{\varphi_0(t,\lambda_n)\cos\lambda_n\xi}{\alpha_n}-\frac{\varphi_0(t,\lambda_n^0)\cos\lambda_n^0\xi}{\alpha_n^0}\right)d\xi,$$

$$I_{N4}(x,t):=\int_0^{\mu^+(t)} A(t,\xi)\sum_{n=1}^{N}\frac{\varphi(x,\lambda_n)\cos\lambda_n\xi}{\alpha_n}d\xi.$$

It is easily found by using (9) and (10)

$$F(x,t)=\sum_{n=1}^{\infty}\left(\frac{\varphi_0(x,\lambda_n)\varphi_0(t,\lambda_n)}{\alpha_n}-\frac{\varphi_0(x,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}\right).$$

Let $f(x)$ be an absolutely continuous function, $f'(0)=f(\pi)=0$. Then using expansion formula (see [21]),

$$\sum_{n=1}^{\infty}\int_0^{\pi} f(t)\rho(t)\frac{\varphi(x,\lambda_n)\varphi(t,\lambda_n)}{\alpha_n}dt=f(x),$$

$$\sum_{n=1}^{\infty}\int_0^{\pi} f(t)\rho(t)\frac{\varphi_0(x,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}dt=f(x). \tag{13}$$

Using (13) we have:

$$\lim_{N\to\infty}\max_{0\le x\le\pi}\left|\int_0^{\pi} f(t)\rho(t)\Phi_N(x,t)dt\right|=$$

$$=\lim_{N\to\infty}\max_{0\le x\le\pi}\left|\int_0^{\pi} f(t)\rho(t)\sum_{n=1}^{N}\left(\frac{\varphi(x,\lambda_n)\varphi(t,\lambda_n)}{\alpha_n}-\frac{\varphi_0(x,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}\right)dt\right|\le$$

$$\le\lim_{N\to\infty}\left\{\max_{0\le x\le\pi}\left|\int_0^{\pi} f(t)\rho(t)\sum_{n=1}^{N}\frac{\varphi(x,\lambda_n)\varphi(t,\lambda_n)}{\alpha_n}dt-f(x)\right|+\right.$$

$$\left.+\max_{0\le x\le\pi}\left|\int_0^{\pi} f(t)\rho(t)\sum_{n=1}^{N}\frac{\varphi_0(x,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}dt-f(x)\right|\right\}=0. \tag{14}$$

We obtain uniformly on $x\in[0,\pi]$

$$\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)I_{N1}(x,t)dt=$$

$$=\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)\sum_{n=1}^{N}\left(\frac{\varphi_0(x,\lambda_n)\varphi_0(t,\lambda_n)}{\alpha_n}-\frac{\varphi_0(x,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}\right)dt=$$

$$=\int_0^{\pi} f(t)\rho(t)F(x,t)dt. \tag{15}$$

It follows from (6) that

$$\cos\lambda\xi=\begin{cases}\varphi_0(\xi,\lambda) & ,\quad \xi<a,\\ \frac{2\alpha}{1+\alpha}\varphi_0(\frac{\xi}{\alpha}+a-\frac{a}{\alpha},\lambda)+\frac{1-\alpha}{1+\alpha}\varphi_0(2a-\xi,\lambda) & ,\quad \xi>a.\end{cases} \tag{16}$$

Taking into account (16) and (13), we get

$$\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)I_{N2}(x,t)dt=$$

$$=\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)\int_0^{\mu^+(x)}A(x,\xi)\sum_{n=1}^{N}\frac{\varphi_0(t,\lambda_n^0)\cos\lambda_n^0\xi}{\alpha_n^0}d\xi dt=$$

$$= \lim_{N\to\infty}\int_0^\pi f(t)\rho(t)\int_0^a A(x,\xi)\sum_{n=1}^N \frac{\varphi_0(t,\lambda_n^0)\varphi_0(\xi,\lambda)}{\alpha_n^0}d\xi dt+$$

$$+\frac{2\alpha}{1+\alpha}\lim_{N\to\infty}\int_0^\pi f(t)\rho(t)\int_a^{\alpha x-\alpha a+a} A(x,\xi)\times$$

$$\times\sum_{n=1}^N \frac{\varphi_0(t,\lambda_n^0)\varphi_0(\frac{\xi}{\alpha}+a-\frac{a}{\alpha},\lambda_n^0)}{\alpha_n^0}d\xi dt+$$

$$+\frac{1-\alpha}{1+\alpha}\lim_{N\to\infty}\int_0^\pi f(t)\rho(t)\int_a^{\alpha x-\alpha a+a} A(x,\xi)\times$$

$$\times\sum_{n=1}^N \frac{\varphi_0(t,\lambda_n^0)\varphi_0(2a-\xi,\lambda_n^0)}{\alpha_n^0}d\xi dt=$$

$$=\int_0^a A(x,\xi)\int_0^\pi f(t)\rho(t)\sum_{n=1}^\infty \frac{\varphi_0(t,\lambda_n^0)\varphi_0(\xi,\lambda_n^0)}{\alpha_n^0}dtd\xi+$$

$$+\frac{2\alpha}{1+\alpha}\int_a^{\alpha x-\alpha a+a} A(x,\xi)\int_0^\pi f(t)\rho(t)\times$$

$$\times\sum_{n=1}^\infty \frac{\varphi_0(t,\lambda_n^0)\varphi_0(\frac{\xi}{\alpha}+a-\frac{a}{\alpha},\lambda_n^0)}{\alpha_n^0}dtd\xi+$$

$$+\frac{1-\alpha}{1+\alpha}\int_a^{\alpha x-\alpha a+a} A(x,\xi)\int_0^\pi f(t)\rho(t)\times$$

$$\times\sum_{n=1}^\infty \frac{\varphi_0(t,\lambda_n^0)\varphi_0(2a-\xi,\lambda_n^0)}{\alpha_n^0}dtd\xi=$$

$$=\int_0^a A(x,\xi)f(\xi)d\xi+\frac{2\alpha}{1+\alpha}\int_a^{\alpha x-\alpha a+a} A(x,\xi)f\left(\frac{\xi}{\alpha}+a-\frac{a}{\alpha}\right)d\xi+$$

$$+\frac{1-\alpha}{1+\alpha}\int_a^{\alpha x-\alpha a+a} A(x,\xi)f\left(2a-\xi\right)d\xi.$$

Substituting $\frac{\xi}{\alpha}+a-\frac{a}{\alpha}\to\xi'$ and $2a-\xi\to\xi''$ we obtain

$$\lim_{N\to\infty}\int_0^\pi f(t)\rho(t)I_{N2}(x,t)dt=\int_0^a A(x,\xi)f\left(\xi\right)d\xi+$$

$$+\frac{2\alpha^2}{1+\alpha}\int_a^x A\left(x,\alpha\xi'-\alpha a+a\right)f\left(\xi'\right)d\xi'+$$

$$+\frac{1-\alpha}{1+\alpha}\int_{-\alpha x+\alpha a+a}^a A(x,2a-\xi'')f\left(\xi''\right)d\xi''.$$

Since $A(x,2a-\xi'')\equiv 0$ when $2a-\xi>\alpha x-\alpha a+a$, we have

$$\lim_{N\to\infty}\int_0^\pi f(t)\rho(t)I_{N2}(x,t)dt=\int_0^a A(x,t)f\left(t\right)dt+$$

$$+\frac{2\alpha^2}{1+\alpha}\int_a^x A\left(x,\alpha t-\alpha a+a\right)f\left(t\right)dt+$$

$$+\frac{1-\alpha}{1+\alpha}\int_0^a A(x,2a-t)f(t)\,dt.$$

Thus, uniformly on $x\in[0,\pi]$:

$$\lim_{N\to\infty}\int_0^\pi f(t)\rho(t)I_{N2}(x,t)dt=\int_0^x\frac{2\rho(t)}{1+\sqrt{\rho(t)}}A(x,\mu^+(t))f(t)\,dt+$$

$$+\int_0^x\frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A(x,2a-t)f(t)\,dt. \tag{17}$$

Using (9), uniformly on $x\in[0,\pi]$

$$\lim_{N\to\infty}\int_0^\pi f(t)\rho(t)I_{N3}(x,t)dt=$$

$$=\lim_{N\to\infty}\int_0^\pi f(t)\rho(t)\int_0^{\mu^+(x)}A(x,\xi)\sum_{n=1}^N\Bigg(\frac{\varphi_0(t,\lambda_n)\cos\lambda_n\xi}{\alpha_n}-$$

$$-\frac{\varphi_0(t,\lambda_n^0)\cos\lambda_n^0\xi}{\alpha_n^0}\Bigg)d\xi dt=$$

$$=\int_0^\pi f(t)\rho(t)\int_0^{\mu^+(x)}A(x,\xi)\sum_{n=1}^\infty\Bigg(\frac{\varphi_0(t,\lambda_n)\cos\lambda_n\xi}{\alpha_n}-$$

$$-\frac{\varphi_0(t,\lambda_n^0)\cos\lambda_n^0\xi}{\alpha_n^0}\Bigg)d\xi dt=$$

$$=\int_0^\pi f(t)\rho(t)\int_0^{\mu^+(x)}A(x,\xi)F_0(\xi,t)d\xi dt. \tag{18}$$

Using the residue theorem and the formula $\frac{\varphi(x,\lambda_n)}{2\lambda_n\alpha_n}=\frac{\psi(x,\lambda_n)}{\dot\Delta(\lambda_n)}$ (see [21]), where $\psi(x,\lambda)$ is the solution of (1) with initial condition $\psi(\pi,\lambda)=0$, $\psi'(\pi,\lambda)=1$ and $\Delta(\lambda)=\varphi(\pi,\lambda)$ is the characteristic function of (1)-(3), $\dot\Delta(\lambda)=\frac{d}{d\lambda}\Delta(\lambda)$, we calculate

$$\lim_{N\to\infty}\int_0^\pi f(t)\rho(t)I_{N4}(x,t)dt=$$

$$=\lim_{N\to\infty}\int_0^\pi f(t)\rho(t)\int_0^{\mu^+(t)}A(t,\xi)\sum_{n=1}^N\frac{\varphi(x,\lambda_n)\cos\lambda_n\xi}{\alpha_n}d\xi dt=$$

$$=2\lim_{N\to\infty}\int_0^\pi f(t)\rho(t)\sum_{|\lambda_n|\le N}\lambda_n\frac{\psi(x,\lambda_n)}{\dot\Delta(\lambda_n)}\int_0^{\mu^+(t)}A(t,\xi)\cos\lambda_n\xi d\xi dt=$$

$$=2\lim_{N\to\infty}\int_0^\pi f(t)\rho(t)\sum_{|\lambda_n|\le N}\underset{\lambda=\lambda_n}{Res}\left[\lambda\frac{\psi(x,\lambda)}{\Delta(\lambda)}\int_0^{\mu^+(t)}A(t,\xi)\cos\lambda\xi d\xi\right]dt=$$

$$=2\lim_{N\to\infty}\int_0^\pi f(t)\rho(t)\frac{1}{2\pi i}\oint_{\Gamma_N}\lambda\frac{\psi(x,\lambda)}{\Delta(\lambda)}\int_0^{\mu^+(t)}A(t,\xi)\cos\lambda\xi d\xi d\lambda dt=$$

$$= \lim_{N\to\infty} \int_0^{\pi} f(t)\rho(t) \frac{1}{\pi i} \oint_{\Gamma_N} \lambda \frac{\psi(x,\lambda)}{\Delta(\lambda)} e^{|Im\lambda|\mu^+(t)} e^{-|Im\lambda|\mu^+(t)} \times$$

$$\times \int_0^{\mu^+(t)} A(t,\xi) \cos\lambda\xi d\xi d\lambda dt =$$

$$= \int_0^{\pi} f(t)\rho(t) \lim_{N\to\infty} \left(\frac{1}{\pi i} \oint_{\Gamma_N} \lambda \frac{\psi(x,\lambda)}{\Delta(\lambda)} e^{|Im\lambda|\mu^+(t)} e^{-|Im\lambda|\mu^+(t)} \times \right.$$

$$\left. \times \int_0^{\mu^+(t)} A(t,\xi) \cos\lambda\xi d\xi d\lambda \right) dt \tag{19}$$

where $\Gamma_N = \{\lambda : |\lambda| = N\}$. Since (see [21])

$$\begin{aligned} \psi(x,\lambda) &= O\left(\frac{e^{|Im\lambda|(\mu^+(\pi)-\mu^+(x))}}{|\lambda|} \right), \quad |\lambda| \to \infty, \\ |\Delta(\lambda)| &\geq C_\delta e^{|Im\lambda|\mu^+(\pi)}, \quad \lambda \in G_\delta, \end{aligned}$$

($G_\delta = \{\lambda : |\lambda - \lambda_n| \geqslant \delta\}$, δ is a sufficiently small positive number) and according to Lemma 1.3.1 from [9]

$$\lim_{|\lambda|\to\infty} \max_{0\le t\le \pi} e^{-|Im\lambda|\mu^+(t)} \left| \int_0^{\mu^+(t)} A(t,\xi)\cos\lambda\xi d\xi d\lambda \right| = 0$$

from the equality (19) we get

$$\lim_{N\to\infty} \int_0^{\pi} f(t)\rho(t) I_{N4}(x,t) dt = 0. \tag{20}$$

Multiplying both sides of (12) by $\rho(x)f(x)$, integrating from 0 to π, tending to limit when $N \to \infty$ and using (14), (15), (17), (18) and (20) we have

$$\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho(t)}} A(x,\mu^+(t)) f(t)\, dt + \int_0^x \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}} A(x,2a-t) f(t)\, dt +$$

$$+ \int_0^{\pi} f(t)\rho(t) F(x,t) dt + \int_0^{\pi} f(t)\rho(t) \int_0^{\mu^+(x)} A(x,\xi) F_0(\xi,t) d\xi dt = 0.$$

Since $f(x)$ can be chosen arbitrarily, we obtain

$$\frac{2\rho(t)}{1+\sqrt{\rho(t)}} A(x,\mu^+(t)) + \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}} A(x,2a-t) + F(x,t) +$$

$$+ \int_0^{\mu^+(x)} A(x,\xi) F_0(\xi,t) d\xi = 0.$$

□

3. Theorem for the Solution of the Inverse Problem

Theorem 2. *For each fixed* $x \in [0, \pi]$ *main equation (8) has a unique solution* $A(x, .) \in L_{2,\rho}(0, \mu^+(x))$.

Proof. We show that for each fixed $x > a$ the equation (8) is equivalent to the equation of the form $(I + B) f = g$ where B is a completely continuous operator, I is an identity operator in the space $L_{2,\rho}(0, \pi)$. (When $x \leq a$ this fact is obvious.)

When $x > a$ rewrite (8) as

$$L_x A(x, .) + K_x A(x, .) = -F(x, .),$$

where

(21)
$$(L_x f)(t) = \begin{cases} f(t) + \frac{1-\alpha}{1+\alpha} f(2a - t) & , \quad t \leq a < x, \\ \frac{2}{1+\alpha} f(\alpha t - \alpha a + a) & , \quad a < t < x. \end{cases}$$

$$(K_x f)(t) = \int_0^{\alpha x - \alpha a + a} f(\xi) F_0(\xi, t) \, d\xi, \quad 0 < t < x.$$

It is sufficient to prove that L_x is invertible, i.e. has a bounded inverse in $L_{2,\rho}(0, \pi)$.

Consider the equation $(L_x f)(t) = \phi(t)$, $\phi(t) \in L_{2,\rho}(0, \pi)$, i.e.

$$\begin{cases} f(t) + \frac{1-\alpha}{1+\alpha} f(2a - t) = \phi(t) & , \quad t \leq a < x, \\ \frac{2}{1+\alpha} f(\alpha t - \alpha a + a) = \phi(t) & , \quad a < t < x. \end{cases}$$

From here it is easily to obtain

$$f(t) = \left(L_x^{-1} \phi\right)(t) = \begin{cases} \phi(t) - \frac{1-\alpha}{2} \phi\left(\frac{-t + \alpha a + a}{\alpha}\right) & , \quad t < a \\ \frac{1+\alpha}{2} \phi\left(\frac{t + \alpha a - a}{\alpha}\right) & , \quad t > a. \end{cases}$$

We show that

$$\|f\|_{L_2} = \left\|L_x^{-1} \phi\right\| \leq C \|\phi\|_{L_2}.$$

In fact,

$$\int_0^\pi |f(t)|^2 \, dt = \int_0^a \left|\phi(t) - \frac{1-\alpha}{2} \phi\left(\frac{-t + \alpha a + a}{\alpha}\right)\right|^2 dt +$$
$$+ \int_a^\pi \left|\frac{1+\alpha}{2} \phi\left(\frac{t + \alpha a - a}{\alpha}\right)\right|^2 dt \leq 2 \int_0^a |\phi(t)|^2 \, dt +$$
$$+2 \left(\frac{1-\alpha}{2}\right)^2 \int_0^a \left|\phi\left(\frac{-t + \alpha a + a}{\alpha}\right)\right|^2 dt +$$
$$+ \left(\frac{1+\alpha}{2}\right)^2 \int_a^\pi \left|\phi\left(\frac{t + \alpha a - a}{\alpha}\right)\right|^2 dt \leq$$
$$\leq 2 \int_0^\pi |\phi(t)|^2 \, dt + \frac{\alpha (1-\alpha)^2}{2} \int_a^{\frac{\alpha a + a}{\alpha}} |\phi(t)|^2 \, dt +$$

$$+\alpha\left(\frac{1+\alpha}{2}\right)^2\int_a^{\frac{\pi+\alpha a-a}{\alpha}}|\phi(t)|^2\,dt.$$

We put $\phi(t)=0$, when $t>\pi$. Then

$$\int_0^\pi |f(t)|^2\,dt\le C\int_0^\pi|\phi(t)|^2\,dt=C\left\|\phi(t)\right\|_{L_{2,\rho}(0,\pi)}.$$

So the operator L_x is invertible in $L_{2,\rho}\left(0,\pi\right)$. Then according to Theorem 3 from [26] (see p. 275) it is sufficient to prove that the equation

$$\frac{2}{1+\sqrt{\rho\left(t\right)}}A(\mu^+(t))+\frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)+$$

$$+\int_0^{\mu^+(x)}A(\xi)F_0(\xi,t)d\xi=0 \tag{22}$$

has only trivial solution $A(t)=0$.

Let $A(t)$ be a non-trivial solution of (22). Then

$$\int_0^x\rho(t)\left(\frac{2}{1+\sqrt{\rho\left(t\right)}}A(\mu^+(t))+\frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\right)^2dt+$$

$$+\int_0^x\rho(t)\left(\frac{2}{1+\sqrt{\rho\left(t\right)}}A(\mu^+(t))+\frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\right)\times$$

$$\times\int_0^{\mu^+(x)}A(\xi)F_0(\xi,t)d\xi dt=0.$$

From (9) we have

$$\int_0^x\rho(t)\left(\frac{2}{1+\sqrt{\rho\left(t\right)}}A(\mu^+(t))+\frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\right)^2dt+$$

$$+\int_0^x\frac{2\rho(t)}{1+\sqrt{\rho\left(t\right)}}A(\mu^+(t))\int_0^{\mu^+(x)}A(\xi)\times$$

$$\times\sum_{n=1}^{\infty}\left(\frac{\varphi_0(t,\lambda_n)\cos\lambda_n\xi}{\alpha_n}-\frac{\varphi_0(t,\lambda_n^0)\cos\lambda_n^0\xi}{\alpha_n^0}\right)d\xi dt+$$

$$+\int_0^x\frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\int_0^{\mu^+(x)}A(\xi)\times$$

$$\times\sum_{n=1}^{\infty}\left(\frac{\varphi_0(t,\lambda_n)\cos\lambda_n\xi}{\alpha_n}-\frac{\varphi_0(t,\lambda_n^0)\cos\lambda_n^0\xi}{\alpha_n^0}\right)d\xi dt=0.$$

Using (7) and (16) we obtain

$$\int_0^x \rho(t)\left(\frac{2}{1+\sqrt{\rho(t)}}A(\mu^+(t))+\frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A(2a-t)\right)^2 dt+$$

$$+\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho(t)}}A(\mu^+(t))\int_0^a A(\xi)\sum_{n=1}^{\infty}\frac{\varphi_0(\xi,\lambda_n)\varphi_0(t,\lambda_n)}{\alpha_n}d\xi dt+$$

$$+\int_0^x \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A(2a-t)\int_0^a A(\xi)\sum_{n=1}^{\infty}\frac{\varphi_0(\xi,\lambda_n)\varphi_0(t,\lambda_n)}{\alpha_n}d\xi dt+$$

$$+\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho(t)}}A(\mu^+(t))\int_a^{\alpha x-\alpha a+a} A(\xi)\times$$

$$\times\sum_{n=1}^{\infty}\frac{2\alpha}{1+\alpha}\frac{\varphi_0\left(\frac{\xi}{\alpha}+a-\frac{a}{\alpha},\lambda_n\right)\varphi_0\left(t,\lambda_n\right)}{\alpha_n}d\xi dt+$$

$$+\int_0^x \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A(2a-t)\int_a^{\alpha x-\alpha a+a} A(\xi)\times$$

$$\times\sum_{n=1}^{\infty}\frac{2\alpha}{1+\alpha}\frac{\varphi_0\left(\frac{\xi}{\alpha}+a-\frac{a}{\alpha},\lambda_n\right)\varphi_0\left(t,\lambda_n\right)}{\alpha_n}d\xi dt+$$

$$+\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho(t)}}A(\mu^+(t))\int_a^{\alpha x-\alpha a+a} A(\xi)\times$$

$$\times\sum_{n=1}^{\infty}\frac{1-\alpha}{1+\alpha}\frac{\varphi_0\left(2a-\xi,\lambda_n\right)\varphi_0\left(t,\lambda_n\right)}{\alpha_n}d\xi dt+$$

$$+\int_0^x \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A(2a-t)\int_a^{\alpha x-\alpha a+a} A(\xi)\times$$

$$\times\sum_{n=1}^{\infty}\frac{1-\alpha}{1+\alpha}\frac{\varphi_0\left(2a-\xi,\lambda_n\right)\varphi_0\left(t,\lambda_n\right)}{\alpha_n}d\xi dt-$$

$$-\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho(t)}}A(\mu^+(t))\int_0^a A(\xi)\sum_{n=1}^{\infty}\frac{\varphi_0(\xi,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}d\xi dt-$$

$$-\int_0^x \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A(2a-t)\int_0^a A(\xi)\sum_{n=1}^{\infty}\frac{\varphi_0(\xi,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}d\xi dt-$$

$$-\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho(t)}}A(\mu^+(t))\int_a^{\alpha x-\alpha a+a} A(\xi)\times$$

$$\times\sum_{n=1}^{\infty}\frac{2\alpha}{1+\alpha}\frac{\varphi_0\left(\frac{\xi}{\alpha}+a-\frac{a}{\alpha},\lambda_n^0\right)\varphi_0\left(t,\lambda_n^0\right)}{\alpha_n^0}d\xi dt-$$

$$-\int_0^x \frac{1-\sqrt{\rho(2a-t)}}{1+\sqrt{\rho(2a-t)}}A(2a-t)\int_a^{\alpha x-\alpha a+a} A(\xi)\times$$

$$\times\sum_{n=1}^{\infty}\frac{2\alpha}{1+\alpha}\frac{\varphi_0\left(\frac{\xi}{\alpha}+a-\frac{a}{\alpha},\lambda_n^0\right)\varphi_0\left(t,\lambda_n^0\right)}{\alpha_n^0}d\xi dt-$$

$$-\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho(t)}} A(\mu^+(t)) \int_a^{\alpha x - \alpha a + a} A(\xi)\times$$

$$\times \sum_{n=1}^{\infty} \frac{1-\alpha}{1+\alpha} \frac{\varphi_0\left(2a-\xi, \lambda_n^0\right) \varphi_0\left(t, \lambda_n^0\right)}{\alpha_n^0} d\xi dt -$$

$$-\int_0^x \frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}} A(2a-t) \int_a^{\alpha x - \alpha a + a} A(\xi)\times$$

$$\times \sum_{n=1}^{\infty} \frac{1-\alpha}{1+\alpha} \frac{\varphi_0\left(2a-\xi, \lambda_n^0\right) \varphi_0\left(t, \lambda_n^0\right)}{\alpha_n^0} d\xi dt = 0.$$

Substituting $\xi \to \frac{\xi}{\alpha} + a - \frac{a}{\alpha}$ in third, fourth, ninth, and tenth double integrals and $\xi \to 2\alpha - \xi$ in fifth, sixth, eleventh and twelfth double integrals we get

$$\int_0^x \rho(t) \left(\frac{2}{1+\sqrt{\rho(t)}} A(\mu^+(t)) + \frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}} A(2a-t) \right)^2 dt +$$

$$+\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho(t)}} A(\mu^+(t)) \int_0^a A(\xi) \sum_{n=1}^{\infty} \frac{\varphi_0(\xi, \lambda_n)\varphi_0(t, \lambda_n)}{\alpha_n} d\xi dt +$$

$$+\int_0^x \frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}} A(2a-t) \int_0^a A(\xi) \sum_{n=1}^{\infty} \frac{\varphi_0(\xi, \lambda_n)\varphi_0(t, \lambda_n)}{\alpha_n} d\xi dt +$$

$$+\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho(t)}} A(\mu^+(t)) \int_a^x A(\mu^+(\xi))\times$$

$$\times \sum_{n=1}^{\infty} \frac{2\alpha^2}{1+\alpha} \frac{\varphi_0(\xi, \lambda_n)\varphi_0(t, \lambda_n)}{\alpha_n} d\xi dt +$$

$$+\int_0^x \frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}} A(2a-t) \int_a^x A(\mu^+(\xi))\times$$

$$\times \sum_{n=1}^{\infty} \frac{2\alpha^2}{1+\alpha} \frac{\varphi_0(\xi, \lambda_n)\varphi_0(t, \lambda_n)}{\alpha_n} d\xi dt +$$

$$+\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho(t)}} A(\mu^+(t)) \int_{-\alpha x + \alpha a + a}^a A(2a-\xi)\times$$

$$\times \sum_{n=1}^{\infty} \frac{1-\alpha}{1+\alpha} \frac{\varphi_0\left(\xi, \lambda_n\right) \varphi_0\left(t, \lambda_n\right)}{\alpha_n} d\xi dt +$$

$$+\int_0^x \frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}} A(2a-t) \int_{-\alpha x + \alpha a + a}^a A(2a-\xi)\times$$

$$\times \sum_{n=1}^{\infty} \frac{1-\alpha}{1+\alpha} \frac{\varphi_0\left(\xi, \lambda_n\right) \varphi_0\left(t, \lambda_n\right)}{\alpha_n} d\xi dt -$$

$$-\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho(t)}} A(\mu^+(t)) \int_0^a A(\xi) \sum_{n=1}^{\infty} \frac{\varphi_0(\xi, \lambda_n^0)\varphi_0(t, \lambda_n^0)}{\alpha_n^0} d\xi dt -$$

$$-\int_0^x \frac{1-\sqrt{\rho\,(2a-t)}}{1+\sqrt{\rho\,(2a-t)}}A(2a-t)\int_0^a A(\xi)\times$$

$$\times\sum_{n=1}^{\infty}\frac{\varphi_0(\xi,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}d\xi dt-$$

$$-\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho\,(t)}}A(\mu^+(t))\int_a^x A(\mu^+(\xi))\times$$

$$\times\sum_{n=1}^{\infty}\frac{2\alpha^2}{1+\alpha}\frac{\varphi_0(\xi,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}d\xi dt-$$

$$-\int_0^x \frac{1-\sqrt{\rho\,(2a-t)}}{1+\sqrt{\rho\,(2a-t)}}A(2a-t)\int_a^x A(\mu^+(\xi))\times$$

$$\times\sum_{n=1}^{\infty}\frac{2\alpha^2}{1+\alpha}\frac{\varphi_0(\xi,\lambda_n^0)\varphi_0(t,\lambda_n^0)}{\alpha_n^0}d\xi dt-$$

$$-\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho\,(t)}}A(\mu^+(t))\int_{-\alpha x+\alpha a+a}^a A(2a-\xi)\times$$

$$\times\sum_{n=1}^{\infty}\frac{1-\alpha}{1+\alpha}\frac{\varphi_0\,(\xi,\lambda_n^0)\,\varphi_0\,(t,\lambda_n^0)}{\alpha_n^0}d\xi dt-$$

$$-\int_0^x \frac{1-\sqrt{\rho\,(2a-t)}}{1+\sqrt{\rho\,(2a-t)}}A(2a-t)\int_{-\alpha x+\alpha a+a}^a A(2a-\xi)\times$$

$$\times\sum_{n=1}^{\infty}\frac{1-\alpha}{1+\alpha}\frac{\varphi_0\,(\xi,\lambda_n^0)\,\varphi_0\,(t,\lambda_n^0)}{\alpha_n^0}d\xi dt=0,$$

from which we have

$$\int_0^x \rho(t)\left(\frac{2}{1+\sqrt{\rho\,(t)}}A(\mu^+(t))+\frac{1-\sqrt{\rho\,(2a-t)}}{1+\sqrt{\rho\,(2a-t)}}A(2a-t)\right)^2 dt+$$

$$+\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho\,(t)}}A(\mu^+(t))\int_0^x \frac{2\rho(\xi)}{1+\sqrt{\rho\,(\xi)}}\times$$

$$\times A(\mu^+(\xi))\sum_{n=1}^{\infty}\frac{\varphi_0\,(\xi,\lambda_n)\,\varphi_0\,(t,\lambda_n)}{\alpha_n}d\xi dt+$$

$$+\int_0^x \frac{1-\sqrt{\rho\,(2a-t)}}{1+\sqrt{\rho\,(2a-t)}}A(2a-t)\int_0^x \frac{2\rho(\xi)}{1+\sqrt{\rho\,(\xi)}}\times$$

$$\times A(\mu^+(\xi))\sum_{n=1}^{\infty}\frac{\varphi_0\,(\xi,\lambda_n)\,\varphi_0\,(t,\lambda_n)}{\alpha_n}d\xi dt+$$

$$+\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho\,(t)}}A(\mu^+(t))\int_0^x \frac{1-\sqrt{\rho\,(2a-\xi)}}{1+\sqrt{\rho\,(2a-\xi)}}\times$$

$$\times A(2a-\xi)\sum_{n=1}^{\infty}\frac{\varphi_0\left(\xi,\lambda_n\right)\varphi_0\left(t,\lambda_n\right)}{\alpha_n}d\xi dt+$$

$$+\int_0^x \frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\int_0^x \frac{1-\sqrt{\rho\left(2a-\xi\right)}}{1+\sqrt{\rho\left(2a-\xi\right)}}\times$$

$$\times A(2a-\xi)\sum_{n=1}^{\infty}\frac{\varphi_0\left(\xi,\lambda_n\right)\varphi_0\left(t,\lambda_n\right)}{\alpha_n}d\xi dt-$$

$$-\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho\left(t\right)}}A(\mu^+(t))\int_0^x \frac{2\rho(\xi)}{1+\sqrt{\rho\left(\xi\right)}}\times$$

$$\times A(\mu^+(\xi))\sum_{n=1}^{\infty}\frac{\varphi_0\left(\xi,\lambda_n^0\right)\varphi_0\left(t,\lambda_n^0\right)}{\alpha_n^0}d\xi dt-$$

$$-\int_0^x \frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\int_0^x \frac{2\rho(\xi)}{1+\sqrt{\rho\left(\xi\right)}}\times$$

$$\times A(\mu^+(\xi))\sum_{n=1}^{\infty}\frac{\varphi_0\left(\xi,\lambda_n^0\right)\varphi_0\left(t,\lambda_n^0\right)}{\alpha_n^0}d\xi dt-$$

$$-\int_0^x \frac{2\rho(t)}{1+\sqrt{\rho\left(t\right)}}A(\mu^+(t))\int_0^x \frac{1-\sqrt{\rho\left(2a-\xi\right)}}{1+\sqrt{\rho\left(2a-\xi\right)}}\times$$

$$\times A(2a-\xi)\sum_{n=1}^{\infty}\frac{\varphi_0\left(\xi,\lambda_n^0\right)\varphi_0\left(t,\lambda_n^0\right)}{\alpha_n^0}d\xi dt-$$

$$-\int_0^x \frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\int_0^x \frac{1-\sqrt{\rho\left(2a-\xi\right)}}{1+\sqrt{\rho\left(2a-\xi\right)}}\times$$

$$\times A(2a-\xi)\sum_{n=1}^{\infty}\frac{\varphi_0\left(\xi,\lambda_n^0\right)\varphi_0\left(t,\lambda_n^0\right)}{\alpha_n^0}d\xi dt=0.$$

Thus we obtain

$$\int_0^x \rho(t)\left(\frac{2}{1+\sqrt{\rho\left(t\right)}}A(\mu^+(t))+\frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\right)^2dt+$$

$$+\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left(\int_0^x \rho(t)\left(\frac{2}{1+\sqrt{\rho\left(t\right)}}A(\mu^+(t))+\frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\right)\varphi_0\left(t,\lambda_n\right)dt\right)^2-$$

$$-\sum_{n=1}^{\infty}\frac{1}{\alpha_n^0}\left(\int_0^x \rho(t)\left(\frac{2}{1+\sqrt{\rho\left(t\right)}}A(\mu^+(t))+\frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\right)\varphi_0\left(t,\lambda_n^0\right)dt\right)^2=0.$$

Using the Parseval's equality

$$\int_0^x \rho(t)f^2(t)dt=\sum_{n=1}^{\infty}\frac{1}{\alpha_n^0}\left(\int_0^x \rho(t)f(t)\varphi_0\left(t,\lambda_n^0\right)dt\right)^2$$

for the function

$$f(t)=\frac{2}{1+\sqrt{\rho\left(t\right)}}A(\mu^{+}(t))+\frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\in L_2(0,x)$$

we have

$$\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left(\int_0^x\rho(t)\left(\frac{2}{1+\sqrt{\rho\left(t\right)}}A(\mu^{+}(t))+\frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\right)\varphi_0\left(t,\lambda_n\right)dt\right)^2=0$$

or

$$\int_0^x\rho(t)\left(\frac{2}{1+\sqrt{\rho\left(t\right)}}A(\mu^{+}(t))+\frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)\right)\varphi_0\left(t,\lambda_n\right)dt=0,\quad n\geqslant 1.$$

Since the system $\left\{\varphi_0\left(t,\lambda_n\right)\right\}_{n\geqslant 1}$ is compete in $L_{2,\rho}\left(0,\pi\right)$, we have

$$\frac{2}{1+\sqrt{\rho\left(t\right)}}A(\mu^{+}(t))+\frac{1-\sqrt{\rho\left(2a-t\right)}}{1+\sqrt{\rho\left(2a-t\right)}}A(2a-t)=0,$$

i.e. $\left(L_xA\right)(t)=0$, where the operator L_x is defined by (21). From invertibility of L_x in $L_{2,\rho}\left(0,\pi\right)$ we get $A(x,.)=0$. □

Theorem 3. *Let* L *and* $\widetilde{L}$ *be two boundary value problems and*

$$\lambda_n=\widetilde{\lambda}_n,\ \alpha_n=\widetilde{\alpha}_n,\ \left(n\in\mathbf{Z}\right).$$

Then

$$q(x)=\widetilde{q}(x)\ x\in[0,\pi].$$

Proof. According to (9) and (10) $F_0(x,t)=\widetilde{F}_0(x,t)$ and $F(x,t)=\widetilde{F}(x,t)$. Then from the main equation (8), we have $A(x,t)=\widetilde{A}(x,t)$. It follows from (5) that $q(x)=\widetilde{q}(x)$ $x\in[0,\pi]$. □

4. Example

Using [27], we can transform the main equation (8) to the following equation:

(23) $$\tilde{A}(x,t)+F(x,t)+\int_0^x\tilde{A}(x,\xi)F(x,\xi)d\xi,\quad 0<t<x,$$

where

(24) $$F(x,t)=\rho(t)\sum_{n=1}^{\infty}\left(\frac{\varphi_0(t,\lambda_n)\varphi_0(x,\lambda_n)}{\alpha_n}+\frac{\varphi_0(t,\lambda_n^0)\varphi_0(x,\lambda_n^0)}{\alpha_n^0}\right)$$

and

$$\tilde{A}(x,t)=\begin{cases} A(x,t) & , \quad 0<t<x \\ A(x,t) & , \quad 0<t<-\alpha x+\alpha a+a \\ A(x,t)+\frac{1-\alpha}{1+\alpha}A(x,2a-t) & , \quad -\alpha x-\alpha a+a<t<a<x \\ \frac{2\alpha^2}{1+\alpha}A(x,\alpha t-\alpha a+a) & , \quad a<t<x. \end{cases}$$

We assume that $\lambda_n=\lambda_n^0=\frac{\pi}{\mu^+(\pi)}\left(n-\frac{1}{2}\right), \quad n\geqslant 1; \quad \alpha_n=\pi, \quad n>1; \quad \alpha_1=\frac{\pi}{2}; \quad \alpha_n^0=\pi, \quad n\geqslant 1$. From the formula (24), we obtain

$$F(x,t)=\frac{1}{\pi}\rho(t)\varphi_0(t,\lambda_1)\varphi_0(x,\lambda_1). \tag{25}$$

Substituting (25) into the main equation (23) we obtain

$$\tilde{A}(x,t)=-\frac{1}{\pi}\rho(t)\varphi_0(t,\lambda_1)\left[\varphi_0(x,\lambda_1)+\int_0^x\tilde{A}(x,\xi)\varphi_0(\xi,\lambda_1)d\xi\right],$$

$$\tilde{A}(x,t)=-\frac{1}{\pi}\rho(t)\varphi_0(t,\lambda_1)L(x), \tag{26}$$

where

$$L(x)=\varphi_0(x,\lambda_1)+\int_0^x\tilde{A}(x,\xi)\varphi_0(\xi,\lambda_1)d\xi. \tag{27}$$

Substituting (26) into (27) we obtain

$$L(x)=\varphi_0(x,\lambda_1)-\frac{1}{\pi}L(x)\int_0^x\rho(\xi)\varphi_0^2(\xi,\lambda_1)d\xi.$$

Using the (6), we calculate the integral into last equation. Then, we have

$$L(x)=\begin{cases} \varphi_0(x,\lambda_1)-\frac{1}{2\pi}L(x)\left(x+\frac{sin2\lambda_1x}{2\lambda_1}\right) & , \quad 0<x<a \\ \varphi_0(x,\lambda_1)-\frac{1}{2\pi}L(x)\Phi(x,\lambda_1) & , \quad a<x<\pi, \end{cases} \tag{28}$$

where

$$\Phi(x,\lambda_1)=\frac{a}{2}+\frac{sin2\lambda_1a}{4\lambda_1}+$$

$$\frac{1}{8}(\alpha+1)^2\left[x-a+\frac{1}{2\lambda_1}\left(\frac{sin2\lambda_1(\alpha x-a\alpha+a)}{\alpha}-sin2\lambda_1a\right)\right]+$$

$$+\frac{1}{2}(\alpha-1)^2\left[x-a-\frac{1}{2\lambda_1}\left(\frac{sin2\lambda_1(-\alpha x+a\alpha+a)}{\alpha}-sin2\lambda_1(2a-x)\right)\right].$$

From (28),

$$L(x)=\begin{cases} \varphi_0(x,\lambda_1)\left[1+\frac{1}{2\pi}\left(x+\frac{sin2\lambda_1x}{2\lambda_1}\right)\right]^{-1} & , \quad 0<x<a \\ \varphi_0(x,\lambda_1)\left[1+\frac{1}{\pi}\Phi(x,\lambda_1)\right]^{-1} & , \quad a<x<\pi, \end{cases} \tag{29}$$

Substituting (29) into (26) we get

$$\tilde{A}(x,t)=\begin{cases} -\frac{1}{\pi}\varphi_0(t,\lambda_1)\varphi_0(x,\lambda_1)\left[1+\frac{1}{2\pi}\left(x+\frac{sin2\lambda_1 x}{2\lambda_1}\right)\right]^{-1} & , \quad 0<x<a \\ -\frac{1}{\pi}\rho(t)\varphi_0(t,\lambda_1)\varphi_0(x,\lambda_1)\left[1+\frac{1}{\pi}\Phi(x,\lambda_1)\right]^{-1} & , \quad a<x<\pi, \end{cases} \tag{30}$$

where $0<t<x$ and $\lambda_1=\frac{\pi}{2\mu^+(\pi)}$. Thus, we obtain the solution of main equation (8). If we use the formula (5) then, we obtain the potential $q(x)$.

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