Mathematical Inverse Problems, Vol. 3, No. 1 (2016), 1-11 ISSN 2381-9634

INVERSION OF A CLASS OF PARABOLIC RADON TRANSFORMS

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Abstract. In recent years, Radon type transforms that integrate functions along families of curves or surfaces, have been intensively studied due to their applications to inverse scattering, synthetic aperture radar (SAR), seismic imaging, medical imaging, etc. In this paper, we consider the transform that integrates a function $f(x)$ in $\mathbb{R}^2$ over a family of parabolas invariant to translation. A new exact inversion formula is presented in the case of fixed axis direction, constant latus rectum and no restriction on the vertex. In addition, we show its relation to the hyperbolic Radon transform and derive its inversion formula. Numerical simulations were performed for the case of the parabolic Radon transform.

Key words and phrases. Radon type transforms; parabolic Radon transforms.

1. Introduction

The Radon transform provides the mathematical tool for a great number of reconstruction problems in medical imaging, seismic imaging, non-destructive testing and other areas. Various generalizations of the Radon transform have been studied [1, 2, 3, 4, 8, 13, 14]. In this paper we introduce the parabolic Radon Transform $\mathcal{R}_l f(\xi,\sigma,l)$ in two dimensions. $\mathcal{R}_l f(\xi,\sigma,l)$ puts into correspondence to a given function $f(x,t)$ its integrals

$$\mathcal{R}_l f(\xi,\sigma,l) = \int_{\mathbb{R}^2} f(x,t)\,\delta(t-\sigma-\frac{(x-\xi)^2}{2l})dxdt.$$

If $\mathcal{R}_l f(\xi,\sigma,l)$ is known for all possible values of its three arguments, then the reconstruction of a function $f(x,t)$ of two variables from $\mathcal{R}_l f$ is an over-determined problem. It is reasonable to expect that one can still uniquely recover f from $\mathcal{R}_l f$ after reducing the degrees of freedom by one. There are many different ways to reduce the dimensions of the data $\mathcal{R}_l f$, e.g. by considering only the data coming from families of parabola with vertices located on x-axis, families of parabola with vertices located on t-axis, or with constant latus rectum l. All of these approaches lead to interesting mathematical problems.

The first work concerning the parabolic Radon transform was done by Cormack [4, 5] who considered the parabolas with rotating central axis around the origin. Some further results presented by Denecker, Van Overloop and Sommend [6] using the parabolic isofocal Radon transform. They showed its relation to the classical Radon transform and derive an exact inversion formula. They extended their idea to arbitrary dimensions case with the aid of Gegenbauer polynomials.

Jollivet, Nguyen and Truong analyzed the related problem and derive an analogue of the central slice theorem for three specific cases of the transform in [11].

Moon studied the related problem where he is considering the family of parabolas with a fixed axis direction and variable latus rectum l called the seismic parabolic Radon transform. He showed that it can be reduced to the line Radon transform and provided an exact inversion formula [17]. Numerical results were provided to demonstrate the accuracy of the suggested algorithm. One can find few works dealing with the Parabolic Radon transform in seismic imaging, also known as the slant slack transform [12, 10, 15, 16, 19, 23].

In this paper we concentrate on the problem of recovering of f from $\mathcal{R}_l f$ data limited to the families of parabolas that depend on the coordinates of the vertex (ξ, σ) for fixed axis direction $\mathbf{t}$ and constant latus rectum l. This problem is similar to the one considered in [11], but we use a different approach to solve it.

The rest of this paper is organized as follows. We first provide the definition of the parabolic Radon transform in $\mathbb{R}^2$ and the notations used in the paper. Section 3 presents new results about inversion of the two dimensional parabolic Radon transform in the case of parabolas with fixed axis direction, constant latus rectum and no restriction on the vertex. The proof is based on the use of the Fourier transform and the theory of integral equations. In section 4, we study the hyperbolic Radon transform and show how we can convert it to a parabolic Radon transform. In section 5, we prove that a more generalized parabolic transforms can be uniquely inverted. Finally, numerical examples of the parabolic Radon transform are presented to illustrate the accuracy and efficiency of the proposed algorithm. Additional remarks are given in the last section with acknowledgments and bibliography.

2. Notations

In this paper, we shall consider the parabolic Radon transform integrating an infinitely differentiable function with compact support $f(x,t)$ along parabolas $\mathcal{P}(\xi, \sigma)$ with fixed axis direction $\mathbf{t}$, constant latus rectum $l > 0$ and no restriction on the vertex (ξ, σ):

$$\mathcal{R}_l f(\xi,\sigma) = \int_{\mathbb{R}^2} f(x,t)\,\delta(t-\sigma-\frac{(x-\xi)^2}{2l})dxdt.$$

The equation of the parabola is

$$t = \sigma + \frac{(x-\xi)^2}{2l}.$$

Let $\widehat{g}(t)$ stands for the Fourier transform with respect to the first variable and can be written as

$$\widehat{g}(t) = \int_{-\infty}^{\infty} g(x,t)e^{-i\omega x}\,dx$$

Furthermore the notation $\otimes$ stands for the convolution of f and g

$$f\otimes\, g(\xi) = \int_{-\infty}^{\infty} f(\xi-s)g(s)\,ds.$$

3. Inversion of the parabolic Radon transform

Theorem 1. $\{$*Fourier slice identity*$\}$
Consider a function $f \in C^\infty(\mathbb{R}^2)$ *supported in* $\mathcal{D} = \{(x,t)\in\mathbb{R}^2 \mid 0 \le x \le \xi_{max}, 0 \le t \le \sigma_{max}\}$. *For* $(\xi,\sigma)\in\mathbb{R}^2$ *we define the parabolic Radon transform by*

$$\mathcal{R}_l f(\xi,\sigma) = \int_{\mathbb{R}^2} f(x,t)\,\delta(t-\sigma-\frac{(x-\xi)^2}{2l})dxdt.$$

An exact solution of the inversion problem for the parabolic Radon transform is given by

$$\widehat{f}(x) = \widehat{\mathcal{R}_l f}(x)\otimes \frac{\omega}{2l\pi}e^{\frac{i\omega x^2}{2l}}.$$

where $\otimes$ *denotes the convolution with respect to the second variable.*

Proof. The equation of the parabola $\mathcal{P}(\xi,\sigma)$ with fixed axis direction $\mathbf{t}$, constant latus rectum $l>0$ and vertex (ξ,σ) is as follows:

$$t = \sigma + \frac{(x-\xi)^2}{2l}.$$

We can define the Parabolic Radon Transform of a given function f as

$$\mathcal{R}_l f(\xi,\sigma) = \int_{\mathbb{R}^2} f(x,t)\,\delta(t-\sigma-\frac{(x-\xi)^2}{2l})dxdt,$$

or

$$\mathcal{R}_l f(\xi,\sigma) = \int_{-\infty}^{\infty} f(x,\sigma+\frac{(x-\xi)^2}{2l})\,dx.$$

Let us apply the Fourier transform with respect to the second variable σ where the Fourier transform is denoted by $\widehat{\mathcal{R}_l f}(\xi)$. So we can write

$$\widehat{\mathcal{R}_l f}(\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, \sigma + \frac{(x-\xi)^2}{2l}) e^{-i\omega\sigma}\, dx\, d\sigma.$$

$$\widehat{\mathcal{R}_l f}(\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, \sigma + \frac{(x-\xi)^2}{2l}) e^{-i\omega\sigma}\, d\sigma\, dx.$$

With appropriate change of variables we obtain

$$\widehat{\mathcal{R}_l f}(\xi) = \int_{-\infty}^{\infty} \widehat{f}(x) e^{\frac{i\omega(x-\xi)^2}{2l}}\, dx \tag{1}$$

where $\widehat{f}(x) = \int_{-\infty}^{\infty} f(x,y)\, e^{-i\lambda y} dy$ is the Fourier transform of $f(x,y)$. It is easy to note that

$$\widehat{\mathcal{R}_l f}(\xi) = \widehat{f}(\xi) \otimes e^{\frac{i\omega\xi^2}{2l}}.$$

where $\otimes$ stands for the convolution defined in the previous section. The convolution can be solved by taking the Fourier transform of both sides with respect to ξ. The convolution becomes a product of two transforms which can be solved explicitly for the transform of f. In fact using

$$\widehat{e^{\frac{i\omega\xi^2}{2l}}} = \sqrt{\frac{2\Pi i l}{\omega}} e^{\frac{-il\tau^2}{2\omega}},$$

we obtain

$$\tilde{\widehat{f}}(\tau) = \tilde{\widehat{\mathcal{R}_l f}}(\tau) \sqrt{\frac{\omega}{2\Pi i l}} e^{\frac{il\tau^2}{2\omega}}.$$

Another direct method of solving the Equation (1) is to use the Fresnel transform ([7, 20]):

$$\widehat{\mathcal{R}_l f}(\xi) = \sqrt{\frac{i2l\pi}{\omega}} F_{\frac{\omega}{2l}}\{\widehat{f}(x)\}$$

where

$$F_\alpha\{f(x)\} = \sqrt{\frac{-i\alpha}{\pi}} \int_{-\infty}^{\infty} f(x') e^{i\alpha(x-x')^2}\, dx'.$$

The Fresnel transform can be inverted in a manner comparable to that of the Fourier transform using the fact that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi(x-x')}\, d\xi = \delta(x-x').$$

In fact, to use this identity we multiply both sides of the Equation (1) by $e^{\frac{i\omega(x'-\xi)^2}{2l}}$ and integrating from $-\infty$ to ∞, we get

$$\int_{-\infty}^{\infty} \widehat{\mathcal{R}_l f}(\xi) e^{\frac{i\omega(x'-\xi)^2}{2l}}\, d\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{f}(x) e^{\frac{i\omega(x-\xi)^2}{2l}}\, e^{\frac{i\omega(x'-\xi)^2}{2l}} dx\, d\xi.$$

Changing the order of integration, we obtain

$$\int_{-\infty}^{\infty} \widehat{\mathcal{R}_l f}(\xi) e^{\frac{i\omega(x'-\xi)^2}{2l}}\, d\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{f}(x) e^{\frac{i\omega(x-\xi)^2}{2l}}\, e^{\frac{i\omega(x'-\xi)^2}{2l}} d\xi\, dx.$$

$$\int_{-\infty}^{\infty} \widehat{\mathcal{R}_l f}(\xi) e^{\frac{i\omega(x'-\xi)^2}{2l}}\, d\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{f}(x) e^{\frac{i\omega}{2l}(x^2-2x\xi-x'^2+2x'\xi)} d\xi\, dx.$$

$$\int_{-\infty}^{\infty} \widehat{\mathcal{R}_l f}(\xi) e^{\frac{i\omega(x'-\xi)^2}{2l}}\, d\xi = \int_{-\infty}^{\infty} \widehat{f}(x)\, e^{\frac{i\omega}{2l}(x^2-x'^2)} dx \int_{-\infty}^{\infty} e^{\frac{i\omega}{2l}(-2x\xi+2x'\xi)}\, d\xi.$$

$$\int_{-\infty}^{\infty} \widehat{\mathcal{R}_l f}(\xi) e^{\frac{i\omega(x'-\xi)^2}{2l}}\, d\xi = \int_{-\infty}^{\infty} \widehat{f}(x)\, e^{\frac{i\omega}{2l}(x^2-x'^2)} \frac{2l\pi}{\omega} \delta(x-x')\, dx.$$

We immediately find a new exact formula for the inversion of the parabolic Radon transform

$$\widehat{f}(x) = \frac{\omega}{2l\pi} \int_{-\infty}^{\infty} \widehat{\mathcal{R}_l f}(\xi) e^{\frac{i\omega(x-\xi)^2}{2l}}\, d\xi.$$

And we finally obtain

$$\widehat{f}(x) = \widehat{\mathcal{R}_l f}(x) \otimes \frac{\omega}{2l\pi} e^{\frac{i\omega x^2}{2l}}.$$

□

4. Inversion of the hyperbolic Radon transform

Corollary 2. *Consider a function* $f \in C^\infty(\mathbb{R}^2)$ *supported in* $\mathcal{D} = \{(x,t) \in \mathbb{R}^2 \mid 0 \le x \le \xi_{max}, 0 \le t \le \sigma_{max}\}$. *For* $(\xi, l) \in \mathbb{R}^2$ *we define the hyperbolic Radon transform by*

$$\mathcal{R}_H f(\xi, l) = \int_{\mathbb{R}^2} f(x,y)\, \delta(y^2 - l^2 - (x-\xi)^2) dxdy.$$

where $\mathcal{H}(\xi, l)$ *is the hyperbola with fixed axis direction* **y** *and vertex* (ξ, l). *An exact solution of the inversion problem for the hyperbolic Radon transform is given by*

$$\widehat{f_h}(x) = \widehat{\mathcal{R}_h}(x) \otimes \frac{\omega}{2l\pi} e^{\frac{i\omega x^2}{2l}}.$$

where $f_h(x,t) = \dfrac{f(x,\sqrt{t})}{2\sqrt{t}}$ *and* $\mathcal{R}_h(\xi, \sigma) = \mathcal{R}_H f(\xi, \sqrt{\sigma})$.

Proof. The equation of the hyperbola $\mathcal{H}(\xi, l)$ with fixed axis direction **y** and vertex (ξ, l) is as follows:

$$y^2 = l^2 + (x - \xi)^2.$$

We can define the hyperbola Radon Transform of a given function f as

$$\mathcal{R}_H f(\xi, l) = \int_{\mathbb{R}^2} f(x, y)\, \delta(y^2 - l^2 - (x - \xi)^2) dx dy.$$

Now changing the variables to $t = y^2$ and $\sigma = l^2$, the equation becomes

$$\mathcal{R}_H f(\xi, \sqrt{\sigma}) = \int_{\mathbb{R}^2} \frac{f(x, \sqrt{t})}{2\sqrt{t}}\, \delta(t - \sigma - (x - \xi)^2) dx dt.$$

we also get

$$\mathcal{R}_h(\xi, \sigma) = \int_{\mathbb{R}^2} f_h(x, t)\, \delta(t - \sigma - (x - \xi)^2) dx dt.$$

where $f_h(x, t) = \dfrac{f(x, \sqrt{t})}{2\sqrt{t}}$ and $\mathcal{R}_h(\xi, \sigma) = \mathcal{R}_H f(\xi, \sqrt{\sigma})$. This change of variables from $\mathcal{R}_H f(\xi, \sqrt{\sigma})$ to $\mathcal{R}_h(\xi, \sigma)$ transforms the hyperbola $\mathcal{H}(\xi, l)$ to a parabola $\mathcal{P}(\xi, \sigma)$. As in the previous section we can invert this integral equation.

$$\widehat{f_h}(x) = \widehat{\mathcal{R}_h}(x) \otimes \frac{\omega}{2l\pi} e^{\frac{i\omega x^2}{2l}}.$$

□

5. Generalized parabolic Radon transform

The linear Radon transform in Euclidean space for an infinitely differentiable function with compact support $f(x, t)$ is defined

$$\mathcal{R}_l f(\xi, \sigma) = \int_{\mathcal{L}(p, \sigma)} f(x, t)\, ds$$

where $\mathcal{L}(p, \sigma)$ stands for the line at distance p from the origin and perpendicular to the vector σ and ds stands for the arc-length measure on this line. Similarly, we can define the generalized Parabolic Radon Transform $\mathcal{R}_l f(\xi, \sigma)$ as follows

$$\mathcal{R}_l f(\xi, \sigma) = \int_{\mathcal{P}(\xi, \sigma)} f(x, t)\, ds$$

where ds denotes the arc-length measure on the parabola $\mathcal{P}(\xi, \sigma)$. We can rewrite this equation as follows

$$\mathcal{R}_l f(\xi, \sigma, l) = \int_{\mathbb{R}^2} f(x, t)\, \delta(t - \sigma - \frac{(x - \xi)^2}{2l}) \sqrt{\frac{(x - \xi)^2}{l^2} + 1}\, dx dt.$$

There is an extra term which does not appear in our definition of the parabolic Radon transform. In this subsection we consider this generalized parabolic Radon transform of an infinitely differentiable function $f(x,t)$ supported inside the $\mathcal{D} = \{(x,t) \in \mathbb{R}^2 \mid 0 \le x \le x_{max}, \epsilon \le t \le \sigma_{max}\}$. We will show that the function can be uniquely recovered from the Radon data $\mathcal{R}_l f$.

Theorem 3. *Consider a function $f \in C^\infty(\mathbb{R}^2)$ supported in $\mathcal{D} = \{(x,t) \in \mathbb{R}^2 \mid 0 \le x \le x_{max}, \epsilon \le t \le \sigma_{max}\}$. For $(\xi,\sigma) \in \mathbb{R}^2$, we define the parabolic Radon transform as follows*

$$\mathcal{R}_l f(\xi,\sigma) = \int_{\mathcal{P}(\xi,\sigma)} f(x,t)\,ds \tag{2}$$

If $\mathcal{R}_l f(\xi,\sigma)$ is known for $\xi \in [-x_{max}, x_{max}]$ and $\sigma \in [-\sigma_{max}, \sigma_{max}]$, then $f(x,t)$ can be uniquely recovered.

Proof. The equation of the parabola $\mathcal{P}(\xi,\sigma)$ with fixed axis direction **t**, constant latus rectum l and vertex (ξ,σ) can be written as:

$$x = \xi \pm \sqrt{2l(t-\sigma)}$$

and

$$ds = \sqrt{\frac{l^2 + 2l(t-\sigma)}{2l(t-\sigma)}}dt$$

Let us rewrite the equation (2) as

$$\mathcal{R}_l f(\xi,\sigma) = \int_\sigma^{\sigma_{max}} f(\xi \pm \sqrt{2l(t-\sigma)}, t)\sqrt{\frac{l^2 + 2l(t-\sigma)}{2l(t-\sigma)}}dt.$$

$$\begin{aligned}\mathcal{R}_l f(\xi,\sigma) = &\int_\sigma^{\sigma_{max}} f(\xi + \sqrt{2l(t-\sigma)}, t)\sqrt{\frac{l^2 + 2l(t-\sigma)}{2l(t-\sigma)}}dt\\ &+ \int_\sigma^{\sigma_{max}} f(\xi - \sqrt{2l(t-\sigma)}, t)\sqrt{\frac{l^2 + 2l(t-\sigma)}{2l(t-\sigma)}}dt.\end{aligned}$$

we apply the Fourier transform to both sides with respect to the first variable

$$\begin{aligned}\int_{-\infty}^{\infty} \mathcal{R}_l f(\xi,\sigma)e^{-i\omega\xi}\,d\xi = &\int_{-\infty}^{\infty}\int_\sigma^{\sigma_{max}} f(\xi + \sqrt{2l(t-\sigma)}, t)\sqrt{\frac{l^2 + 2l(t-\sigma)}{2l(t-\sigma)}}dte^{-i\omega\xi}\,d\xi\\ &+ \int_{-\infty}^{\infty}\int_\sigma^{\sigma_{max}} f(\xi - \sqrt{2l(t-\sigma)}, t)\sqrt{\frac{l^2 + 2l(t-\sigma)}{2l(t-\sigma)}}dte^{-i\omega\xi}\,d\xi.\end{aligned}$$

we change the order of integration and with a change a variable $u = \xi \pm \sqrt{2l(t-\sigma)}$ we obtain

$$\widehat{\mathcal{R}_l f}(\sigma) = \int_{\sigma}^{\sigma_{max}} \int_{-\infty}^{\infty} f(u,t) \sqrt{\frac{l^2 + 2l(t-\sigma)}{2l(t-\sigma)}} e^{-i\omega\xi}\, du\, dt$$

$$+ \int_{\sigma}^{\sigma_{max}} \int_{-\infty}^{\infty} f(u,t) \sqrt{\frac{l^2 + 2l(t-\sigma)}{2l(t-\sigma)}} e^{-i\omega\xi}\, du\, dt.$$

Let the Fourier transform

$$\widehat{f_\omega(t)} = \int_{-\infty}^{\infty} f(u,t) e^{-i\omega u} du$$

we can write

$$\widehat{\mathcal{R}_l f}(\sigma) = \int_{\sigma}^{\sigma_{max}} \widehat{f_\omega(t)} \sqrt{\frac{l^2 + 2l(t-\sigma)}{2l(t-\sigma)}} e^{-i\sqrt{2l(t-\sigma)}}\, dt$$

$$+ \int_{\sigma}^{\sigma_{max}} \widehat{f_\omega(t)} \sqrt{\frac{l^2 + 2l(t-\sigma)}{2l(t-\sigma)}} e^{i\sqrt{2l(t-\sigma)}}\, dt.$$

$$\widehat{\mathcal{R}_l f}(\sigma) = 2\int_{\sigma}^{\sigma_{max}} \widehat{f_\omega(t)} \sqrt{\frac{l^2 + 2l(t-\sigma)}{2l(t-\sigma)}} \cos(\sqrt{2l(t-\sigma)})\, dt$$

$$\widehat{\mathcal{R}_l f}(\sigma) = \int_{\sigma}^{\sigma_{max}} \widehat{f_\omega(t)} \frac{K(t,\sigma)}{\sqrt{t-\sigma}} dt$$

where the kernel

$$K(t,\sigma) = 2\sqrt{\frac{l^2 + 2l(t-\sigma)}{2l}} \cos(\sqrt{2l(t-\sigma)}).$$

This equation is a Volterra integral equation of the first kind with weakly singular kernel (see [21] page 361,[24, 25]). The kernel $K(t,\sigma)$ is continuous in its arguments along with the first order partial derivatives on $(0, \sigma_{max})$ and $K(\sigma,\sigma) \neq 0$. Hence this type of equations have a unique solution and there is a standard approach to find the solution through a resolvent kernel using Picard's process of successive approximations (see [2, 3]).

□

6. Numerical implementation

The proposed inversion formula for the parabolic Radon transform is illustrated with some numerical examples. Similarly to [8, 9], we present the results of these experiments for smooth phantoms supported within a square $\Omega = [-1,1] \times [-1,1]$ and defined by functions of the form

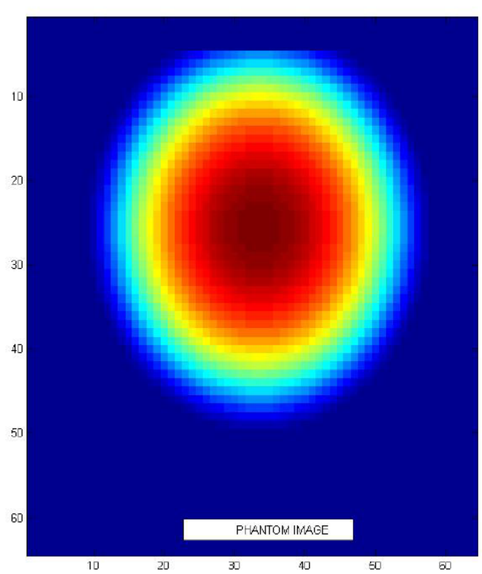
$$f(x,y) = \begin{cases} exp\left\{\dfrac{-r^2}{r^2 - [(x-x_c)^2 + (y-y_c)^2]}\right\} & \text{if } (x-x_c)^2 + (y-y_c)^2 < r^2, \\ 0 & \text{otherwise .} \end{cases}$$

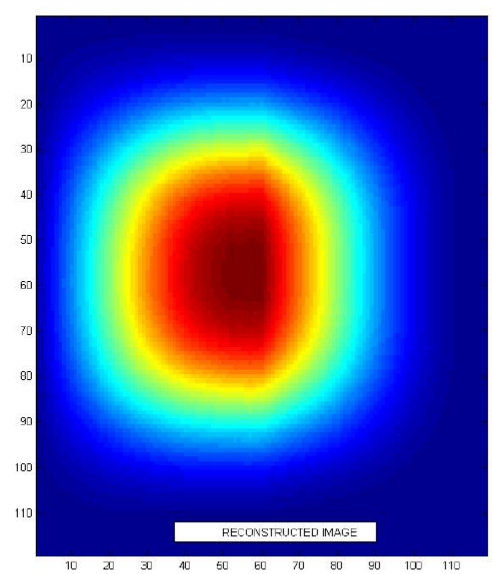
In all experiments we used a fixed $l = 0.05$. The parabolic Radon transform was computed numerically from an $N \times N$ sampled function f. We parametrized the parabola of integration with two parameters (ξ, σ) the coordinates of the vertex. The generated data are the values of the line integrals over the family of parbolas $\mathcal{R}_l f(\xi, \sigma)$.

Once the projection data has been generated, we reconstruct important features (e.g. boundaries) of the original phantom. The reconstruction algorithm is based on an approximate algorithm presented in [18]. The important steps of the back-projection are outlined below:

(1) Perform the projection data for each vertex $(\xi_i, \sigma j)$.
(2) Reformulate $\dfrac{\omega}{2l\pi} e^{\frac{i\omega x^2}{2l}}$ as a $\dfrac{\sin(x)}{x}$ function based on an approximate algorithm presented in [18].
(3) Convolve the result with the projections $\mathcal{R}_l f(\xi_i, \sigma_j)$. The reconstruction image is an approximation of $f(x, y)$.

Figure 6 shows the results for a phantom with $r = 0.25$ and the center $(x_c, y_c) = (0.2, 0.3)$ using discretization $N = 60$.





7. Additional remarks

(1) The smoothness and decay conditions for f in both inversion formulae are not optimized. The formulae may hold with weaker requirements, e.g. for f in Schwartz space, or compactly supported functions that have only a few orders of derivatives.

(2) The generalization of the inversion formula to higher dimensions is an interesting problem in its own. The author plan to address this problem in future research.

Acknowledgements

The work of the author was supported by the American University of Sharjah (AUS) research grant FRG3, UAE.

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