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A NEW COMPUTATIONAL METHOD FOR FUZZY LAPLACE TRANSFORMS BY THE DIFFERENTIAL TRANSFORM METHOD

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ABSTRACT. In this paper, the differential transform method is applied to compute the fuzzy Laplace transform of fuzzy functions. The approach exhibits simplicity in that, unlike the usual method which necessitates integration, it only entails easy differentiation and a few elementary operations. Illustrative examples are provided to demonstrate the applicability and efficiency of the technique.
Key words and phrases. Fuzzy integral; differential transform method; Fuzzy Laplace transform.

1. INTRODUCTION

One of the interesting transforms in the theory of fuzzy differential equations (FDEs) is Laplace transforms. The fuzzy Laplace transform method solves FDEs and corresponding fuzzy initial and boundary value problems. In this way, the fuzzy Laplace transforms reduce the problem of solving a FDEs to an algebraic problem. This switching from operations of calculus to algebraic operations on transforms is called operational calculus, a very important area of applied mathematics, and for the engineer, the fuzzy Laplace transform method is practically the most important operational method. The fuzzy Laplace transform also has the advantage that it solves problems directly without determining a general solution in the first and obtaining non homogeneous differential equations in the second. One can see some useful papers about fuzzy Laplace transforms in [2, 14, 15]. Also, there exist some recently published papers with some modifications about application of fuzzy Laplace transforms to solve fuzzy differential equation [18, 19].Most phenomena in the real world, are described through nonlinear equations which have been attractive to scientists fairly recently, much attention has been focused on a powerful technique, namely, the differential transform method (DTM), for treating functional equations. The concept of the DTM was first suggested by Zhou in his study on electrical circuits and the equations arising [19]. Later on, it was extended to solve a wide span of equations, as well as systems of equations, including algebraic, differential, integral, and integro-differential ones [22]. In a nutshell, the DTM recursively furnishes analytical Taylor series solutions and the literature abounds with a multitude applications of it (see [3, 7, 9, 12, 21, 26]). It is the aim of this paper to present a simple integration-free scheme to yield the fuzzy Laplace transform of any desired function by invoking the DTM. As is known, the standard routine for the derivation of fuzzy Laplace transforms inherits an improper integration which may, in certain cases, be not analytically tractable. However, in contrast, the proposed s-traightforward approach merely requires easy fuzzy differentiations and algebraic operations. The reliability and efficiency of the new method are well illustrated with a number of examples. In this work, we intend to use DTM for computing the fuzzy Laplace transforms. The paper is organized as follows: In section 2, we present the basic notions of fuzzy number, fuzzy valued function, fuzzy derivative, fuzzy integral and, fuzzy Laplace transform. In section 3, Definitions and operations of the differential transform method given. In section 4, DTM to compute the fuzzy Laplace transform given. Conclusions are drawn in section 5.

2. PRELIMINARIES

We represent an arbitrary fuzzy number by an ordered pair function $(\underline{u}(r), \overline{u}(r))$, which satisfies the following requirements [11]:

a: $\underline{u}(r)$ is abounded monotonic increasing left continuous function,

b: $\overline{u}(r)$ is abounded monotonic decreasing left continuous function,

c: $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$. We recall that for $a < b < c$ which $a, b, c \in R$, the triangular fuzzy number $u = (a, b, c)$ are determined by a, b, c such that $\underline{u}(r) = a + (b - c)r$ and $\overline{u}(r) = c - (c - b)r$ are the endpoints of the r-level sets, for all $r \in [0, 1]$. Let E be the set of all fuzzy number on $\Re$.

For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $k > 0$, we define addition $u \oplus v$, subtraction $u \ominus v$ and scalar multiplication by k as [10].

a) Addition:

$u \oplus v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,

b) subtraction:

$u \ominus v = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r))$,

c) scalar multiplication:

$$k \odot u = \begin{cases} (k\underline{u}, k\overline{u}) & k \geq 0 \\ & , \\ (k\overline{u}, k\underline{u}), & k < 0 \end{cases}$$

if $k = -1$ then $k \odot u = -u$.

Definition 1. [17] For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$, we show the Hausdorff distance between u and v by $D(u, v)$, and take $D : E \times E \longrightarrow \Re_+ \cup (0)$. Also, we know (E, D) is a complete metric space, thus:

$$D(u,v) = \sup_{0\leq r\leq 1} \{\max[|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|]\},$$

We have following traits for Hausdorff distance; per $u, v, e, f \in E$ and all $k \in \Re$:
i) $D(u + e, v + e) = D(u, v)$,
ii) $D(ku, kv) = |k|D(u, v)$,
iii) $D(u + v, e + f) \leq D(u, e) + D(v, f)$.

Definition 2. [10] Let $f : \Re \longrightarrow E$ be a fuzzy-valued function. If for arbitrary fixed $x_0 \in \Re$ and $\epsilon > 0$, a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow D(f(x), f(x_0)) < \epsilon,$$

f is said to be continuous.

Definition 3. [20] A mapping $f : \Re \times E \longrightarrow E$ is called continuous at point $(t_0, x_0) \in \Re \times E$ provided for any fixed $r \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists an $\delta(\epsilon, r) > 0$, such that

$$D([f(t, x)]_r, [f(t_0, x_0)]_r) < \epsilon,$$

whenever $|t - t_0| < \delta$ and $D([x]_r, [x_0]_r) < \delta(\epsilon, r)$ for all $t \in \Re,\ x \in E$.

Theorem 1. [23] Let $f(x)$ be a fuzzy value function on $[a, b]$ and it is represented by $(\underline{f}(x, r), \overline{f}(x, r))$ for $r \in [0, 1]$, assume $\overline{f}(x, r)$ are Riemann-integrable on $[a, b]$ for every $b \geq a$ and assume. There are two positive values $\underline{M}(r)$ and $\overline{M}(r)$ such that

$$\int_a^b |\underline{f}(x, r)|dx \leq \underline{M}(r)\ ,$$

and

$$\int_a^b |\overline{f}(x, r)|dx \leq \overline{M}(r)\ ,$$

for every $b \geq a$.

Then $f(x)$ is improper fuzzy Riemann-integrable on $[a, \infty)$ and is a fuzzy number. furthermore, we have:

$$\int_a^\infty f(x)dx = (\int_a^\infty \underline{f}(x)dx, \int_a^\infty \overline{f}(x)dx)\ .$$

Proposition 1. [24] If $f(x)$ and $g(x)$ are fuzzy value functions and fuzzy Riemann-integrable on $[a, \infty)$ then $f(x) + g(x)$ is fuzzy Riemann-integrable on $[a, \infty)$.

Moreover , we have:

$$\int_l (f(x) \oplus g(x))dx = \int_l f(x)dx \oplus \int_l g(x)dx\ .$$

It is well-known that the H-derivative (differentiability in the sense of Hukuhara) for fuzzy mappings was initially introduced by Puri and Ralescu [16] and it is based in the H-difference of sets, as follows.

Definition 4. Suppose $x, y \in E$. If there exists $z \in E$ such that $x = y \oplus z$, then z is called the

H-difference of x and y, and it is denoted by $x -^h y$.
In this paper, the sign "$-^h$" always stands for H-difference and also note that $x -^h y \neq x \ominus y$. We consider the following definition which was introduced by Bede et al. [5].
Definition 5. Suppose $f : (a,b) \to E$ and $x_0 \in (a,b)$. We say that f is strongly generalized differential at x_0 (Bede et al. [6]) if there exists an element $f'(x_0) \in E$, such that:
a) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) -^h f(x_0)$, $\exists f(x_0) -^h f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f(x_0 + h) -^h f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0) -^h f(x_0 - h)}{h} = f'(x_0)$$

or
b) for all $h > 0$ sufficiently small, $\exists f(x_0) -^h f(x_0 + h)$, $\exists f(x_0 - h) -^h f(x_0)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f(x_0) -^h f(x_0 + h)}{-h} = \lim_{h\to 0} \frac{f(x_0 - h) -^h f(x_0)}{-h} = f'(x_0)$$

or
c) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) -^h f(x_0)$, $\exists f(x_0 - h) -^h f(x_0)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f(x_0 + h) -^h f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0 - h) -^h f(x_0)}{-h} = f'(x_0)$$

or
d) for all $h > 0$ sufficiently small, $\exists f(x_0) -^h f(x_0 + h)$, $\exists f(x_0) -^h f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f(x_0) -^h f(x_0 + h)}{-h} = \lim_{h\to 0} \frac{f(x_0) -^h f(x_0 - h)}{h} = f'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively)
Theorem 2. [8] Let $f : R \longrightarrow E$ be a function and denote $f(t) = (\underline{f}(t,r), \overline{f}(t,r))$ for each $r \in [0,1]$. Then
(1) If f is (i)-differentiable, then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and $f'(t) = (\underline{f}'(t,r), \overline{f}'(t,r))$,
(2) If f is (ii)-differentiable, then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and $f'(t) = (\overline{f}'(t,r), \underline{f}'(t,r))$.

Definition 6. Let $f(x)$ be continuous fuzzy-value function. Suppose that $f(x) \odot e^{-sx}$ is improper fuzzy Riemann integrable on $[0, \infty)$, then $\int_0^\infty f(x) \odot e^{-sx} dx$ is called fuzzy Laplace transforms and is denoted as:

$$L[f(x)] = \int_0^\infty f(x) \odot e^{-sx} dx \; (s > 0 \; and \; integer).$$

From theorem 1, we have:

$$\int_0^\infty f(x) \odot e^{-sx} dx = \left(\int_0^\infty \underline{f}(x,r) e^{-sx} dx, \int_0^\infty \overline{f}(x,r) e^{-sx} dx \right),$$

also by using the definition of classical Laplace transform:

$$\underline{F}(s,r) = L[\underline{f}(x,r)] = \int_0^\infty \underline{f}(x,r) e^{-sx} dx,$$

and

$$\overline{F}(s,r) = L[\overline{f}(x,r)] = \int_0^\infty \overline{f}(x,r) e^{-sx} dx,$$

then, we follow:

$$L[f(x)] = F(s) = (L[\underline{f}(x,r), L[\overline{f}(x,r)]).$$

3. Definitions and operations of the differential transform method

For the convenience of the reader, we concisely review the fundamentals of the DTM in this section. The one-dimensional differential transform of the function $u(x)$ is defined by the following formula

(1) $$U(k) = \frac{1}{k!} [\frac{d^k u(x)}{dx^k}]_{x=0}$$

where $u(x)$ and $U(k)$ are the original and transformed functions, respectively. Accordingly, the inverse differential transform of $U(k)$ is specified as follows:

(2) $$u(x) = \sum_{k=0}^{\infty} \{U(k) x^k\}.$$

From Eqs.(1) and (2) we get

(3) $$u(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{d^k u(x)}{dx^k} |_{x=0}$$

The preceding equation is indicative of the great simplicity in computation of differential inverse transform. This is opposite to the case for analytical inversion of Laplace transforms. For the sake of brevity, we do not include definitions/theorems pertaining to n-dimensional differential transforms and recommend that the interested reader sees [7, 21, 26]. Some fundamental operations of the one-dimensional differential transform are listed in Table1. Their proofs are easily followed from the definitions (1),(2) and are fully covered in [3, 9, 12]. Note that represents the Kronecker delta function and m is a non-negative integer. For conciseness of notation, we abbreviate the direct and inverse differential transform operators as DT and DT^{-1}, respectively, in this paper.

Table 1

Some operations of the one-dimensional differential transform

Original function	Transformed function
$u(x) \pm v(x)$	$U(k) \pm V(k)$
$\alpha u(x)$	$\alpha U(k)$
$\frac{d^m u(x)}{dx^m}$	$\frac{(k+m)!}{k!}U(k+m)$
$u(x)v(x)$	$\sum_{r=0}^{k}\{u(r)v(k-r)\}$
e^{ax}	$\frac{a^k}{k!}$
$f(x) = \int_{x_0}^{x} u(t)dt$	$F(k) = \frac{u(k-1)}{k}, k \geq 1, F(0) = 0$
$f(x) = \int_{x_0}^{x} g(t)u(t)dt$	$F(k) = \sum_{l=0}^{k-1}\{G(l)\frac{U(k-l-1)}{k}\}, k \geq 1, F(0) = 0$
x^m	$\delta(k-m) = \begin{cases} 1, & k = m, \\ 0, & k \neq m. \end{cases}$
$sin(ax)$	$\frac{a^k}{k!}sin(\frac{k\pi}{2}) = \begin{cases} 0, & k \in even, \\ \frac{a^k(-1)^{\frac{k-1}{2}}}{k!}, & k \in odd. \end{cases}$
$cos(ax)$	$\frac{a^k}{k!}cos(\frac{k\pi}{2}) = \begin{cases} \frac{a^k(-1)^{\frac{k}{2}}}{k!}, & k \in even, \\ 0, & k \in odd. \end{cases}$

Theorem 3. Let m be a finite positive integer such that $m \geq 1$ and $s > 0$, $s \in R$. It holds that

$$\lim_{x\to\infty}\sum_{k=1}^{\infty}\{\frac{k^m}{k!}(-sx)^k)\} = 0$$

Proof.

We use mathematical induction for the proof of the above-described theorem.

Step1 : Let $m = 1$.

$$\sum_{k=1}^{\infty}\{\frac{k^m}{k!}(-sx)^k\} = \sum_{k=1}^{\infty}\{\frac{(-sx)^k}{(k-1)!}\} = \sum_{k=0}^{\infty}\{\frac{(-sx)^{k+1}}{(k)!}\} = (-sx)\sum_{k=0}^{\infty}\{\frac{(-sx)^k}{(k)!}\} = (-sx)e^{-sx}$$

thus,

$$\lim_{x\to\infty}\sum_{k=1}^{\infty}\{\frac{k^m}{k!}(-sx)^k)\} = \lim_{x\to\infty}(-sx)e^{-sx} = 0.$$

Since

$$\sum_{k=1}^{\infty}\{\frac{k}{k!}(-sx)^k)\} = (-sx)e^{-sx}\ then\ (-sx)\sum_{k=1}^{\infty}\{\frac{k}{k!}(-sx)^k)\} = (-sx)^2e^{-sx},$$

it is also concluded that

$$\lim_{x\to\infty}(-sx)\sum_{k=1}^{\infty}\{\frac{k}{k!}(-sx)^k)\} = 0$$

Step 2: Let $m = 2$.

$$\sum_{k=1}^{\infty}\{\frac{k^2}{k!}(-sx)^k\} = \sum_{k=1}^{\infty}\{\frac{k}{(k-1)!}(-sx)^k\} = \sum_{k=0}^{\infty}\{\frac{k+1}{k!}(-sx)^{k+1}\} =$$

$$(-sx)(\sum_{k=0}^{\infty}\{\frac{k}{k!}(-sx)^k\} + \sum_{k=0}^{\infty}\{\frac{(-sx)^k}{k!}\}) = (-sx)(-sxe^{-sx} + e^{-sx})$$

Consequently,

$$\lim_{x\to\infty}\sum_{k=1}^{\infty}\{\frac{k^2}{k!}(-sx)^k)\} = \lim_{x\to\infty}(-sx)(-sxe^{-sx} + e^{-sx}) = 0.$$

Also,

$$\lim_{x\to\infty}(-sx)^2(-sxe^{-sx} + e^{-sx}) = 0.$$

Step 3: Assume that for$m = 1$ to i it holds true that $\lim_{x\to\infty}\sum_{k=1}^{\infty}\{\frac{k^i}{k!}(-sx)^k)\} = 0$ and $\lim_{x\to\infty}(-sx)\sum_{k=1}^{\infty}\{\frac{k^i}{k!}(-sx)^k)\} = 0$.

In this context, we have to prove that

$$\lim_{x\to\infty}\sum_{k=1}^{\infty}\{\frac{k^{i+1}}{k!}(-sx)^k)\} = 0.$$

We have

$$\sum_{k=1}^{\infty}\{\frac{k^{i+1}}{k!}(-sx)^k)\} = \sum_{k=1}^{\infty}\{\frac{k^i}{(k-1)!}(-sx)^k\} = \sum_{k=0}^{\infty}\{\frac{(k+1)^i}{k!}(-sx)^{(k+1)}\}$$

$$= (-sx)\sum_{k=0}^{\infty}\{\frac{(k+1)^i}{k!}(-sx)^k\} = (-sx)\sum_{k=0}^{\infty}\{\frac{(-sx)^k}{k!}\} + \binom{i}{1}(-sx)\sum_{k=0}^{\infty}\{\frac{k}{k!}(-sx)^k\}$$

$$+\binom{i}{2}(-sx)\sum_{k=0}^{\infty}\{\frac{k^2}{k!}(-sx)^k\} + \cdots + (-sx)\sum_{k=0}^{\infty}\{\frac{k^i}{k!}(-sx)^k\}$$

and thus, $\lim_{x\to\infty}\sum_{k=1}^{\infty}\{\frac{k^{m+1}}{k!}(-sx)^k)\} = 0$.

Now, as the argument in step 3 is valid for $i = 2$ (shown in steps 1 and 2), we conclude that for $m = 3$ the theorem is satisfied. Subsequently, by incrementally raising i from 3 to m (and calling step 3 each time), the theorem is proved recursively for any desired value of m.

4. DTM TO COMPUTE THE FUZZY LAPLACE TRANSFORM

Consider the fuzzy first-order differential equations $\frac{du}{dx} - su = f(x)$, $u(0) = 0$, $s > 0$. The solution of equation can be expressed in the following integration:

$$u(x)\mu(x) = \int f(x)\mu(x)dx, \tag{4}$$

where

$$\mu(x) = e^{\int -sdx},$$

If the crisp function $\mu(x)$ is continuous in the metric D , its definite integral exists. Furthermore

$$\underline{u}(x,r)e^{-sx}|_0^\infty = \int_0^\infty \underline{f}(x,r)e^{-sx}dx, \tag{5}$$

and

$$\overline{u}(x,r)e^{-sx}|_0^\infty = \int_0^\infty \overline{f}(x,r)e^{-sx}dx. \tag{6}$$

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach. More details about properties of the fuzzy integral are given in [13, 24].

Generally the integral of Eq.(5) and Eq.(6) are complicated and can not be expressed in term of elementary functions nor conveniently tabulated in open literature. However, this method is a powerful tool to calculate such difficult fuzzy integrals.From the definition of fuzzy Laplace transform, we can write

$$L(\underline{f}(x,r) = [\int_{x_0}^x \underline{f}(x,r)e^{-sx}dx]_0^\infty \tag{7}$$

$$L(\overline{f}(x,r) = [\int_{x_0}^x \overline{f}(x,r)e^{-sx}dx]_0^\infty \tag{8}$$

Taking direct and inverse differential transforms with respect to x on both sides of Eq. (7),(8) simultaneously, we obtain

$$L(\underline{f}(x,r) = DT^{-1}\{DT\{[\int_{x_0}^x \underline{f}(x,r)e^{-sx}dx]_0^\infty\}\} \tag{9}$$

$$L(\overline{f}(x,r) = DT^{-1}\{DT\{[\int_{x_0}^x \overline{f}(x,r)e^{-sx}dx]_0^\infty\}\} \tag{10}$$

or equally

$$L(\underline{f}(x,r) = [DT^{-1}\{DT\{\int_{x_0}^x \underline{f}(x,r)e^{-sx}dx\}\}]_0^\infty \tag{11}$$

$$(12) \qquad L(\overline{f}(x,r) = [DT^{-1}\{DT\{\int_{x_0}^{x} \overline{f}(x,r)e^{-sx}dx\}\}]_0^{\infty}$$

Now, the differential transform of the inner functional integral can be evaluated by virtue of the last, or its previous, operation displayed in Table1. Afterwards, the expression is formally inverted with the help of (2). Subsequently, some elementary arithmetic operations and series rearrangements are performed to extract an exponential term. A theorem which is stated below may come in handy within this step. Finally, applying the bounds (zero and infinity) to the finally resulting expression gives the Fuzzy Laplace transform of $(\underline{f}(x,r), \overline{f}(x,r))$.

Theorem 4. Let $f(x)$ and $g(x)$ be continuous fuzzy-value functions and suppose that α is constant, then

$$L[f(x) \oplus (\alpha \odot g(x))] = F(s) \oplus (\alpha \odot G(s)).$$

Proof. According to (9), (10) If $\alpha > 0$, we have

$$\begin{aligned} L[\underline{f}(x,r) + \alpha\underline{g}(x,r)] &= e^{-sx}DT^{-1}\{DT(\{[\int_{x_0}^{x} \underline{g}(x,r)dx]_0^{\infty}\} + \alpha\{[\int_{x_0}^{x} \underline{f}(x,r)dx]_0^{\infty}\})\} \\ &= DT^{-1}\{DT\{[\int_{x_0}^{x} \underline{f}(x,r)e^{-sx}dx]_0^{\infty}\}\} + \alpha DT^{-1}\{DT\{[\int_{x_0}^{x} \underline{g}(x,r)e^{-sx}dx]_0^{\infty}\}\} \\ &= L[\underline{f}(x,r)] + \alpha L[\underline{g}(x,r)] \end{aligned}$$

also, similarly

$$\begin{aligned} L[\overline{f}(x,r) + \alpha\overline{g}(x,r)] &= e^{-sx}DT^{-1}\{DT(\{[\int_{x_0}^{x} \overline{g}(x,r)dx]_0^{\infty}\} + \alpha\{[\int_{x_0}^{x} \overline{f}(x,r)dx]_0^{\infty}\})\} \\ &= DT^{-1}\{DT\{[\int_{x_0}^{x} \overline{f}(x,r)e^{-sx}dx]_0^{\infty}\}\} + \alpha DT^{-1}\{DT\{[\int_{x_0}^{x} \overline{g}(x,r)e^{-sx}dx]_0^{\infty}\}\} \\ &= L[\overline{f}(x,r)] + \alpha L[\overline{g}(x,r)] \end{aligned}$$

therefore

$$L[\underline{f}(x,r) \oplus (\alpha \odot \underline{g}(x,r))] = \underline{F}(s) \oplus (\alpha \odot \underline{G}(s)).$$

If $\alpha < 0$, we have

$$\begin{aligned} L[\underline{f}(x,r) + \alpha\overline{g}(x,r)] &= e^{-sx}DT^{-1}\{DT(\{[\int_{x_0}^{x} \underline{f}(x,r)dx]_0^{\infty}\} + \alpha\{[\int_{x_0}^{x} \overline{g}(x,r)dx]_0^{\infty}\})\} \\ &= DT^{-1}\{DT\{[\int_{x_0}^{x} \underline{f}(x,r)e^{-sx}dx]_0^{\infty}\}\} + \alpha DT^{-1}\{DT\{[\int_{x_0}^{x} \overline{g}(x,r)e^{-sx}dx]_0^{\infty}\}\} \\ &= L[\underline{f}(x,r)] + \alpha L[\overline{g}(x,r)] \end{aligned}$$

also, similarly

$$L[\overline{f}(x,r)+\alpha \underline{g}(x,r)] = L[\overline{f}(x,r)] + \alpha L[\underline{g}(x,r)]$$

therefore

$$L[f(x) \oplus (\alpha \odot g(x))] = (L[\underline{f}(x,r)] + \alpha L[\overline{g}(x,r)], L[\overline{f}(x,r)] + \alpha L[\underline{g}(x,r)]) = F(s) \oplus (\alpha \odot G(s)).$$

Theorem 5. Let $f(x)$ be continuous fuzzy-value function and $L[f(x)] = F(s)$. Suppose that a is constant, then

$$L[f(ax)] = \frac{1}{a} \odot F(\frac{s}{a}).$$

Proof. According to (9), (10), If $a > 0$, we have

$$\begin{aligned} L[\underline{f}(ax,r)] =& [DT^{-1}\{DT\{\int_{ax_0}^{ax} \underline{f}(ax,r)e^{\frac{-s}{a}ax}dx\}\}]_0^\infty \\ =& \frac{1}{a}[DT^{-1}\{DT\{\int_{x_0}^{x} \underline{f}(ax,r)e^{\frac{-s}{a}ax}dx\}\}]_0^\infty \\ =& \frac{1}{a} \odot \underline{F}(\frac{s}{a}) \end{aligned}$$

also, similarly

$$\begin{aligned} L[\overline{f}(ax,r)] =& [DT^{-1}\{DT\{\int_{ax_0}^{ax} \overline{f}(ax,r)e^{\frac{-s}{a}ax}dx\}\}]_0^\infty \\ =& \frac{1}{a}[DT^{-1}\{DT\{\int_{x_0}^{x} \overline{f}(ax,r)e^{\frac{-s}{a}ax}dx\}\}]_0^\infty \\ =& \frac{1}{a} \odot \overline{F}(\frac{s}{a}) \end{aligned}$$

therefore

$$L[\underline{f}(ax,r), \overline{f}(ax,r)] = \frac{1}{a} \odot [\underline{F}(\frac{s}{a}), \overline{F}(\frac{s}{a})]$$

The same trend holds for $a < 0$.

Theorem 6. Let $f(x)$ be continuous fuzzy-value function and $L[f(x)] = F(s)$, then $L[f(x) \odot e^{ax}] = F(s-a)$ for $s > a$.

Proof. According to (9), (10), we have

$$L[\underline{f}(x,r)\odot e^{ax}] = e^{ax}(DT^{-1}\{DT\{[\int_{x_0}^{x}\underline{f}(x,r)e^{-sx}dx]_0^{\infty}\}\})$$
$$= DT^{-1}\{DT\{[\int_{x_0}^{x}\underline{f}(x,r)e^{-(s-a)x}dx]_0^{\infty}\}\}$$
$$= \underline{F}(s-a,r)$$

also, similarly

$$L[\overline{f}(x,r)\odot e^{ax}] = \overline{F}(s-a,r),$$

therefore

$$L[f(x)\odot e^{ax}] = (\underline{F}(s-a,r),\overline{F}(s-a,r)) = F(s-a).$$

4.1. **Application.** In this section, we compute the fuzzy Laplace transforms of a number of different functions by the proposed method for the sake of exemplification.

Example 1. Suppose $f(x,r) = (2r-5,-3r)e^{ax}$ and $s > a$. We have:

$$\underline{F}(s,r) = [DT^{-1}\{DT\{\int_{x_0}^{x}(2r-5)\underline{f}(x,r)e^{ax}e^{-sx}dx\}\}]_0^{\infty}$$
$$= (2r-5)[DT^{-1}\{DT\{\int_{x_0}^{x}\underline{f}(x,r)e^{-(s-a)x}dx\}\}]_0^{\infty}$$
$$= (2r-5)[DT^{-1}\{\frac{(-s+a)^{k-1}}{k\times(k-1)!}\}]_{x=0}^{\infty}$$
$$= (2r-5)[\sum_{k=0}^{\infty}\{\frac{(-s+a)^{k-1}x^k}{k\times(k-1)!}]_{x=0}^{\infty}$$
$$= \frac{2r-5}{s-a}[\sum_{k=0}^{\infty}\{\frac{(-s+a)^k x^k}{k!}]_{x=0}^{\infty}$$
$$= \frac{2r-5}{s-a}[e^{-(s-a)}]_{x=0}^{\infty} = \frac{2r-5}{s-a}$$

and

$$\begin{aligned}
\overline{F}(s,r) =&[DT^{-1}\{DT\{\int_{x_0}^{x}(-3r)\overline{f}(x,r)e^{ax}e^{-sx}dx\}\}]_0^{\infty}\\
&= (-3r)[DT^{-1}\{DT\{\int_{x_0}^{x}\overline{f}(x,r)e^{-(s-a)x}dx\}\}]_0^{\infty}\\
&= [DT^{-1}\{\frac{(-s+a)^{k-1}}{k\times(k-1)!}\}]_{x=0}^{\infty}\\
&= (-3r)[\sum_{k=0}^{\infty}\{\frac{(-s+a)^{k-1}x^k}{k\times(k-1)!}]_{x=0}^{\infty}\\
&= \frac{-3r}{s-a}[\sum_{k=0}^{\infty}\{\frac{(-s+a)^{k}x^k}{k!}]_{x=0}^{\infty}\\
&= \frac{-3r}{s-a}[e^{-(s-a)}]_{x=0}^{\infty} = \frac{-3r}{s-a}
\end{aligned}$$

therefore

$$F(s) = (2r-5,-3r)\frac{1}{s-a}.$$

Example 2. Suppose $f(x,r) = (r^2+r, 4-r-r^3)x^m$. We have:

$$\begin{aligned}
L(\underline{f}(x,r)) = L((r^2+r)\underline{x}^m) &= [DT^{-1}\{DT\{\int_{x_0}^{x}\underline{f}(x,r)e^{-sx}dx\}\}]_0^{\infty}\\
&= [\sum_{k=0}^{\infty}\{DT\{\int_{x_0}^{x}(r^2+r)x^m e^{-sx}dx\}x^k\}\}]_0^{\infty} \qquad (13)\\
&= (r^2+r)[\sum_{k=0}^{\infty}\{DT\{\int_{x_0}^{x}x^m e^{-sx}dx\}x^k\}]_0^{\infty}.
\end{aligned}$$

From the properties included in Table 1, especially the one in the last line, it is obtained that

$$DT\{\int_{x_0}^{x}(r^2+r)x^m e^{-sx}dx\} = \begin{cases} 0, & k=0\\ \sum_{l=0}^{k-1}\{\frac{(r^2+r)\delta(k-m-l-1)}{k}\frac{(-s)^l}{l!}\}, & k\geq 1\end{cases} \qquad (14)$$

Substituting Eq. (13) into Eq. (12) gives

$$L((r^2+r)x^m) = 0 + [\sum_{k=1}^{\infty}\{\sum_{l=0}^{k-1}\{\frac{(r^2+r)\delta(k-m-l-1)(-s)^l}{kl!}\}\}x^k]_{x=0}^{\infty} \qquad (15)$$

The definition of Kronecker delta function forces

$$
(16)\quad \begin{aligned}
L((r^2+r)x^m) =& [\sum_{k=0}^{\infty}\{\frac{(r^2+r)}{k}\frac{(-s)^{k-1-m}}{(k-1-m)!}x^k\}]_{x=0}^{\infty} \\
&= [\frac{(r^2+r)(-1)^{m+1}}{s^{m+1}}\sum_{k=0}^{\infty}\{\frac{(k-1)(k-2)(k-3)\dots(k-m)x^k}{k!}\}]_{x=0}^{\infty} \\
&= \frac{(r^2+r)(-1)^{m+1}}{s^{m+1}}([\sum_{k=1}^{\infty}\{\frac{(-1)(-2)\dots(-m)(-s)^k}{k!}x^k\}]_{x=0}^{\infty} \\
&+ \sum_{i=0}^{m}\{a_i \lim_{x} \to \infty \sum_{k=1}^{\infty}\{\frac{k!}{k!}x^k\})
\end{aligned}
$$

where ai denotes the constant coefficient pertaining to the ith power of k in the polynomial generated by $(k\text{-}1)(k\text{-}2)\dots(k\ \text{-}m)$. According to the lemma, the last summation in Eq. (15) fades out and therefore

$$
(17)\quad \begin{aligned}
L((r^2+r)x^m) =& \frac{(r^2+r)(-1)^{m+1}}{s^{m+1}}([\sum_{k=1}^{\infty}\{\frac{(-1)(-2)\dots(-m)(-s)^k}{k!}x^k\}]_{x=0}^{\infty} \\
&= \frac{(r^2+r)(-1)^{2m+1}m!}{s^{m+1}}[\sum_{k=1}^{\infty}\{\frac{k!}{k!}x^k\}]_{x=0}^{\infty} \\
&= \frac{(r^2+r)(-m!)}{s^{m+1}}[\sum_{k=1}^{\infty}\{\frac{k!}{k!}x^k\}]_{x=0}^{\infty} \\
&= \frac{(r^2+r)(-m!)}{s^{m+1}}[e^{-sx}-1]_{x=0}^{\infty} = \frac{(r^2+r)(m!)}{s^{m+1}}
\end{aligned}
$$

also, similarly

$$
L(\overline{f}(t,r)) = L((4-r-r^3)x^m) = \frac{(4-r-r^3)m!}{s^{m+1}}
$$

therefore

$$
\begin{aligned}
L(\underline{f}(x,r),\overline{f}(t,r)) =& L((r^2+r)x^m,(4-r-r^3)x^m) \\
&= (\frac{(r^2+r)m!}{s^{m+1}},\frac{(4-r-r^3)m!}{s^{m+1}}) \\
&= (r^2+r, 4-r-r^3)\frac{m!}{s^{m+1}}
\end{aligned}
$$

Example 3. Suppose $f(t,r) = (2r+1, 4-r)sin(at)$. We have:

$$L(\underline{f}(t,r)) = L((2r+1)sin(at)) = [DT^{-1}\{DT\{\int_{x_0}^{x}(2r+1)sin(at)e^{-sx}dx\}\}]_0^\infty$$
$$= [DT^{-1}\{\sum_{l=0}^{k-1}\{\frac{a^l(2r+1)}{l!}sin(\frac{l\pi}{2})\frac{\frac{(-s)^{k-l-1}}{(k-l-1)!}}{k}\}\}]_{x=0}^\infty$$
$$= [\sum_{k=0}^{\infty}\{\sum_{l=1,3,5}^{k-1}\{\frac{a^l(2r+1)}{l!}\frac{(k-1)(k-2)\dots(k-l)}{s^{l+1}}(-1)^{\frac{l-1}{2}-l-1}\}\frac{x^k(-s)^k}{k!}\}]_0^\infty$$
$$= [\sum_{k=0}^{\infty}\{\sum_{l=1,3,5}^{k-1}\{\frac{a^l(2r+1)}{l!}\frac{(k-1)(k-2)\dots(k-l)}{s^{l+1}}(-1)^{\frac{l-3}{2}}\}\frac{x^k(-s)^k}{k!}\}]_0^\infty$$
$$= [\sum_{k=0}^{\infty}\{\frac{a(2r+1)}{s^2}(k-1)\frac{(-sx)^k}{k!} - \frac{a^3(2r+1)}{s^4}\frac{(k-1)(k-2)(k-3)}{3!}\frac{(-sx)^k}{k!} + \dots\}]_{x=0}^\infty$$

According to the lemma and knowing that for $m \geq 1$, $\lim_{x\to\infty}\sum_{k=1}^{\infty}\{\frac{k^{m+1}}{k!}(-sx)^k)\} = 0$. We obtain a simplified form:

$$[\sum_{k=0}^{\infty}\{\frac{-a(2r+1)}{s^2}\frac{(-sx)^k}{k!} + \frac{a^3(2r+1)}{s^4}\frac{(-sx)^k}{k!} + \dots\}]_{x=0}^\infty$$
$$= [(\frac{-a(2r+1)}{s^2} + \frac{a^3(2r+1)}{s^4} - \dots)e^{-sx}]_{x=0}^\infty$$
$$= \frac{-a(2r+1)}{s^2} - \frac{a^3(2r+1)}{s^4} + \frac{a^5(2r+1)}{s^5} + \dots$$

An infinite geometric progression is obtained and the summation of its first n components, when n inclines toward infinity, is found as follows:

$$L((2r+1)sin(at)) = \frac{\frac{(2r+1)a}{s^2}}{1+\frac{a^2}{s^2}} = \frac{(2r+1)a}{s^2+a^2},$$

also, similarly

$$L(\overline{f}(t,r)) = L((4-r)sin(at)) = \frac{(4-r)a}{s^2+a^2},$$

therefore

$$L(\underline{f}(t,r), \overline{f}(t,r)) = (\frac{(2r+1)a}{s^2+a^2}, \frac{(4-r)a}{s^2+a^2}).$$

5. Conclusion

A simple routine for the computation of analytical fuzzy Laplace transforms through a differential transform method has been proposed. The salient point of the new technique lies in it being integration-free, unlike the usual method. The suggested method can be regarded as a powerful computational tool which merely requires simple differentiations and. In this paper, we have verified its efficiency in computing the fuzzy Laplace transform, which is the most important fuzzy integral transform. The classic calculation of fuzzy Laplace transform involves a computation of an infinite range definite integral. Instead, the proposed method based on DTM uses differentiations, so it can be used as an alternative. We have also verified the basic properties of the fuzzy Laplace transform in this new frame work.Using these properties and the presented examples, it would be easy to calculate the fuzzy Laplace transform of a large number of fuzzy functions.

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