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EXPANDING THE APPLICABILITY TIKHONOV'S REGULARIZATION FOR NONLINEAR ILL-POSED PROBLEMS

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ABSTRACT. In [7] the authors presented a cubically convergent Two Step Newton Tikhonov Method (TSNTM) to approximate a solution of an ill-posed equation. In the present paper we show how to expand the applicability of (TSNTM). In particular, we present a semilocal convergence analysis of (TSNTM) under: weaker hypotheses, weaker convergence criteria, tighter error estimates on the distances involved and at least as precise information on the location of the solution.

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1. Introduction

In this study we consider the task of approximately solving the nonlinear ill-posed operator equation

$$F(x) = f, \tag{1.1}$$

where $F : D(F) \subseteq X \to Y$ is a nonlinear operator between the Hilbert spaces X and Y. Let $B_r(x)$ and $\overline{B_r(x)}$, stand respectively, for the open and closed ball in X with center x and radius $r > 0$. Let $\langle ., . \rangle$ denote the inner product and $\|.\|$ denote the corresponding norm. It is assumed that (1.1) has a solution, namely $\hat{x}$, i.e., $F(\hat{x}) = f$. We assume throughout that $f^\delta \in Y$ are the available data such that $\|f - f^\delta\| \leq \delta$. Hence the problem of computing of $\hat{x}$ from equation $F(x) = f^\delta$ is ill-posed (irregular) problem. In such a case, it is necessary either to pass to regularized analogues of these methods on the basis of the iterative regularization principle ([1], [12], [16], [8], [9], [11], [14], [18], [19], [10], [22]-[27]) or to apply these iterative processes to the regularized equation([15], [16], [29])

$$S_\alpha(x) := F'(x)^*(F(x) - f^\delta) + \alpha(x - x_0) = 0 \tag{1.2}$$

for some fixed and appropriately chosen regularization parameter α and initial guess x_0 (see [29]). It is known that the solution u_α^δ of the equation (1.2) is an approximation of $\hat{x}$ provided $\alpha > 0$ is chosen properly (see [28]).

Observe that the operator $S_\alpha(x)$ in (1.2) is the gradient of the Tikhonov ([17], [13], [29]) functional

$$\Phi(x) = \frac{1}{2}\|F(x) - f^\delta\|^2 + \alpha\|x - x_0\|^2.$$

In [29], Vasin considered the iterative method

$$u_\alpha^{k+1} = u_\alpha^k - [F'(u_\alpha^k)^* F'(u_\alpha^k) + \bar{\alpha}I]^{-1} S_\alpha(u_\alpha^k) \tag{1.3}$$

and its modified variant in the form

$$u_\alpha^{k+1} = u_\alpha^k - [F'(u_\alpha^0)^* F'(u_\alpha^0) + \bar{\alpha}I]^{-1} S_\alpha(u_\alpha^k) \tag{1.4}$$

with $\bar{\alpha} > \alpha$ for approximation of the solution u_α^δ of the equation (1.2). The results in [29], was proved using the following conditions

$$\|F'(x)\| \leq N_1, \;\; \|F'(x) - F'(y)\| \leq N_2\|x - y\| \tag{1.5}$$

where $N_1 > 0,\ N_2 > 0$ are constants. Recently, in [30], Vasin and George considered a modified variant of (1.4), i.e., the iteration

$$u_\alpha^{k+1} = u_\alpha^k - [A^*A + \beta I]^{-1}[A^*(F(u_\alpha^k) - y^\delta) + \alpha(u_\alpha^k - u_0)], \;\; u_\alpha^0 = u_0, \tag{1.6}$$

where $A := F'(u_0)$, $\alpha > 0$ is the regularization parameter and β is a constant. In [30], instead of Lipschitz condition (1.5), the following center Lipschitz condition is used.

ASSUMPTION 1.1. *Suppose there exists constants $L_0 > 0$ such that for all $x \in B(x_0, r) \subseteq D(F)$ and $w \in X$, there exists elements $\varphi(x, x_0, w) \in X$ such that*

$$[F'(x) - F'(x_0)]w = F'(x_0)\varphi(x, x_0, w), \;\; \|\varphi(x, x_0, w)\| \leq L_0\|x - x_0\|\|w\|.$$

In [7], the authors considered the following Two Step Newton Tikhonov Method(TSNTM) defined by:

$$y_{n,\alpha}^\delta = x_{n,\alpha}^\delta - R_\alpha(x_{n,\alpha}^\delta)^{-1}[A_0^*(F(x_{n,\alpha}^\delta) - y^\delta) + \alpha(u_{n,\alpha}^\delta - x_0)] \tag{1.7}$$

and

$$x_{n+1,\alpha}^\delta = y_{n,\alpha}^\delta - R_\alpha(x_{n,\alpha}^\delta)^{-1}[A_0^*(F(y_{n,\alpha}^\delta) - y^\delta) + \alpha(y_{n,\alpha}^\delta - x_0)], \tag{1.8}$$

where $x_{0,\alpha}^\delta = x_0,\ R_\alpha(x) := (A_0^*A_x + \alpha I),\ A_x := F'(x),\ A_0 = F'(x_0)$ and $\alpha > 0$ is the regularization parameter and proved that $x_{n,\alpha}^\delta$ converges cubically to the solution x_α^δ of

$$A_0^*F(x_\alpha^\delta) + \alpha(x_\alpha^\delta - x_0) = A_0^*y^\delta \tag{1.9}$$

and that x_α^δ is an approximation of $\hat{x}$..

The semilocal convergence analysis was based on the following conditions which has been used extensively in the study of iterative procedures for solving ill-posed problems [31], [33], [36].

(C1) There exists a constant $L > 0$ such that for each $x, u \in D(F)$ and $v \in X,$ there exists an element $P(x, u, v) \in X$ satisfying

$$[F'(x) - F'(u)]v = F'(u)P(x, u, v), \quad \|P(x, u, v)\| \leq L\|v\|\|x - u\|.$$

In the present paper, we extend the convergence domain of (TSNTM) under weaker sufficient semilocal convergence criteria. Moreover, the upper bounds on the distances $\|x_{n+1,\alpha}^\delta - x_{n,\alpha}^\delta\|, \|x_{n,\alpha}^\delta - x_\alpha^\delta\|$ are tighter and the information on the location of the solution x_α^δ at least as precise (see Section 3).

There are cases when Lipschitz-type condition $(C1)$ is violated (see Section 4) but the weaker central-Lipschitz condition in Assumption 1.1 is satisfied. Note that $L_0 \leq L$ hold in general and $\frac{L}{L_0}$ can be arbitrarily large [1]-[6].

In section 2 we provide a semilocal convergence analysis for (TSNTM) using Assumption 1.1 instead of $(C1)$. We shall refer to [30], [16] for some of the proofs omitted in this study.

2. Semilocal convergence of (TSNTM)

In this section we present the semilocal convergence of (TSNTM) using Assumption 1.1. In due course we shall make use of the following lemma extensively.

LEMMA 2.1. *Let $L_0 r < 1$ and $u \in B_r(x_0)$. Then $(A_0^* A_u + \alpha I)$ is invertible:*

(i)

$$(A_0^* A_u + \alpha I)^{-1} = [I + (A_0^* A_0 + \alpha I)^{-1} A_0^* (A_u - A_0)]^{-1} (A_0^* A_0 + \alpha I)^{-1}$$

and

(ii)

$$\|(A_0^* A_u + \alpha I)^{-1} A_0^* A_0\| \leq \frac{1}{1 - L_0 r},$$

where $A_u := F'(u)$.

Proof. Note that by Assumption 1.1, we have

$$\begin{aligned}
\|(A_0^* A_0 + \alpha I)^{-1} A_0^* (A_u - A_0)\| &= \sup_{\|v\| \leq 1} \|(A_0^* A_0 + \alpha I)^{-1} A_0^* (A_u - A_0)v\| \\
&= \sup_{\|v\| \leq 1} \|(A_0^* A_0 + \alpha I)^{-1} A_0^* A_0 \Phi(u, x_0, v)\| \\
&\leq L_0 \|u - x_0\| \leq L_0 r < 1.
\end{aligned}$$

So $I+(A_0^*A_0+\alpha I)^{-1}A_0^*(A_u-A_0)$ is invertible. Now (i) follows from the following relation

$$A_0^*A_u+\alpha I=(A_0^*A_0+\alpha I)[I+(A_0^*A_0+\alpha I)^{-1}A_0^*(A_u-A_0)].$$

To prove (ii), observe that by Assumption 1.1, we have

$$\begin{aligned}
\|(A_0^*A_u+\alpha I)^{-1}A_0^*A_0\| &= \sup_{\|v\|\leq 1}\|(A_0^*A_u+\alpha I)^{-1}A_0^*A_0v\| \\
&= \sup_{\|v\|\leq 1}\|[I+(A_0^*A_0+\alpha I)^{-1}A_0^*(A_u-A_0)]^{-1} \\
&\quad (A_0^*A_0+\alpha I)^{-1}A_0^*A_0v\| \\
&\leq \frac{1}{1-L_0r}\|(A_0^*A_0+\alpha I)^{-1}A_0^*A_0v\|] \\
&\leq \frac{1}{1-L_0r}.
\end{aligned}$$

This completes the proof.

We need to introduce some sequences and parameters:

(2.1) $$e_{n,\alpha}^\delta := \|y_{n,\alpha}^\delta - x_{n,\alpha}^\delta\|, \qquad \forall n=0,1,\cdots,$$

for $\delta_0<(17-12\sqrt{2})\sqrt{\alpha_0}$ for some $\alpha_0>0$ and $\|x_0-\hat{x}\|\leq\rho$,

(2.2) $$\rho\leq\frac{\sqrt{1+2L_0(17-12\sqrt{2}-\frac{\delta_0}{\sqrt{\alpha_0}})}-1}{L_0}=\rho_0.$$

Let

(2.3) $$b_\rho=\frac{L_0}{2}\rho^2+\rho+\frac{\delta_0}{\sqrt{\alpha_0}},$$

(2.4) $$r=\frac{1}{L_0}\frac{2b_\rho}{1-b_\rho+\sqrt{(1-b_\rho)^2-32b_\rho}},$$

(2.5) $$\gamma_\rho=\frac{1}{1-L_0r}[\frac{L_0}{2}\rho^2+\rho+\frac{\delta_0}{\sqrt{\alpha_0}}],$$

and

(2.6) $$p=2L_0r, q=2p^2.$$

Note that r is well defined, since $\frac{p}{2} < 1, q \in (0,1)$ and $b_\rho \in (0, 17 - 12\sqrt{2}]$. Also note that $r > \frac{1}{2\sqrt{2}L_0}$ and hence $8L_0^3 r^3 > L_0 r$, so we have

$$\begin{aligned}
\frac{1+L_0 r}{1-8L_0^2 r^2}\gamma_\rho &= \frac{1+L_0 r}{(1-8L_0^2 r^2)(1-L_0 r)} b_\rho \\
&= \frac{1+L_0 r}{1-8L_0^2 r^2 + (8L_0^3 r^3 - L_0 r)} b_\rho \\
&\leq \frac{1+L_0 r}{1-8L_0^2 r^2} b_\rho = \frac{1+\frac{p}{2}}{1-q} b_\rho = L_0 r. \qquad (2.7)
\end{aligned}$$

In order for us to simplify the notation, let x_n, y_n and e_n, stand, respectively for $x_{n,\alpha}^\delta, y_{n,\alpha}^\delta$ and $e_{n,\alpha}^\delta$. If we simply use the needed Assumption 1.1 instead of $(C1)$ we arrive at:

LEMMA 2.2. *Suppose that Assumption 1.1 holds and γ_ρ is given by (2.5). Then, the following assertion holds*

$$e_0 \leq \gamma_\rho$$

Proof. Using (2.1), (2.2), (2.3) and $(C1)''$ we obtain in turn that

$$\begin{aligned}
e_0 = \|y_0 - x_0\| &= \|R_\alpha(x_0)^{-1} A_0^*(F(x_0) - f^\delta)\| \\
&= \|R_\alpha(x_0)^{-1} A_0^*[F(x_0) - F(\hat{x}) - F'(x_0)(x_0 - \hat{x}) \\
&\quad + F'(x_0)(x_0 - \hat{x}) + F(\hat{x}) - f^\delta]\| \\
&= \|R_\alpha(x_0)^{-1} A_0^*[\int_0^1 (F'(x_0 + t(\hat{x} - x_0)) - F'(x_0)) dt (x_0 - \hat{x}) \\
&\quad + F'(x_0)(x_0 - \hat{x}) + F(\hat{x}) - f^\delta]\| \\
&\leq \frac{1}{1 - L_0 r}[\| \int_0^1 \Phi(x_0 + t(\hat{x} - x_0), x_0, x_0 - \hat{x})\| + \|x_0 - \hat{x}\| \\
&\quad + \|R_\alpha(x_0)^{-1} A_0^*(F(\hat{x}) - f^\delta)\|] \\
&\leq \frac{1}{1 - L_0 r}[\frac{L_0}{2}\|x_0 - \hat{x}\|^2 + \|x_0 - \hat{x}\| + \frac{1}{\alpha}\|F(\hat{x}) - f^\delta\|] \\
&\leq \frac{1}{1 - L_0 r}[\frac{L_0}{2}\rho^2 + \rho + \frac{\delta}{\sqrt{\alpha}}] \\
&\leq \frac{1}{1 - L_0 r}[\frac{L_0}{2}\rho^2 + \rho + \frac{\delta_0}{\sqrt{\alpha_0}}] = \gamma_\rho.
\end{aligned}$$

The proof of the Lemma is complete. With the notion introduced so far we can present the semilocal convergence analysis of (TSNTM) using the next three results.

THEOREM 2.3. *Suppose that Assumption 1.1 holds and $\delta \in (0, \delta_0]$. Then, the following assertions hold*

(a) $\|x_n - y_n\| \leq p\|y_{n-1} - x_{n-1}\| = pe_{n-1}$,

(b) $\|x_n - x_{n-1}\| \leq (1 + \frac{p}{2})e_{n-1}$,

(c) $e_n \leq qe_{n-1}$.

Proof. Using (1.7) and (1.8) we get that

$$
\begin{aligned}
x_n - y_{n-1} &= y_{n-1} - x_{n-1} - R_\alpha(x_{n-1})^{-1}[A_0^*(F(y_{n-1}) - F(x_{n-1})) \\
&\quad +\alpha(y_{n-1} - x_{n-1})] \\
&= R_\alpha(x_{n-1})^{-1}[R_\alpha(x_{n-1})(y_{n-1} - x_{n-1}) \\
&\quad -A_0^*(F(y_{n-1}) - F(x_{n-1})) - \alpha(y_{n-1} - x_{n-1})] \\
&= R_\alpha(x_{n-1})^{-1}A_0^* \int_0^1 \{F'(x_{n-1}) - F'(x_{n-1} + t(y_{n-1} - x_{n-1}))\} \\
&\quad \times(y_{n-1} - x_{n-1})dt \\
&= R_\alpha(x_{n-1})^{-1}A_0^* \int_0^1 \{F'(x_{n-1}) - F'(x_0) + F'(x_0) \\
&\quad -F'(x_{n-1} + t(y_{n-1} - x_{n-1}))\}(y_{n-1} - x_{n-1})dt. \qquad (2.8)
\end{aligned}
$$

In view of Assumption 1.1 and (2.8) we have that

$$
\begin{aligned}
\|x_n - y_{n-1}\| &\leq \frac{1}{1 - L_0 r}[\| \int_0^1 \Phi(x_{n-1}, x_0, y_{n-1} - x_{n-1})dt\| \\
&\quad +\| \int_0^1 \Phi(x_{n-1} + t(y_{n-1} - x_{n-1}), x_0, x_{n-1} - y_{n-1})dt\|] \\
&\leq \frac{1}{1 - L_0 r}[L_0[\|x_{n-1} - x_0\| \\
&\quad + \int_0^1 \|x_{n-1} - x_0 + t(y_{n-1} - x_{n-1})\|dt]\|y_{n-1} - x_{n-1}\|] \\
&\leq \frac{1}{1 - L_0 r}[2L_0 r\|y_{n-1} - x_{n-1}\| \\
&= p\|y_{n-1} - x_{n-1}\|] = pe_{n-1}.
\end{aligned}
$$

This proves (a). Now (b) follows from (a) and the triangle inequality;

$$\|x_n - x_{n-1}\| \leq \|x_n - y_{n-1}\| + \|y_{n-1} - x_{n-1}\|.$$

To prove (c) we first use (1.7) and (1.8) to obtain in turn the identity

$$
\begin{aligned}
y_n - x_n &= x_n - y_{n-1} - R_\alpha(x_n)^{-1}[A_0^*(F(x_n) - f^\delta) + \alpha(x_n - x_0)] \\
&\quad + R_\alpha(x_{n-1})^{-1}[A_0^*(F(y_{n-1}) - f^\delta) + \alpha(y_{n-1} - x_0)] \\
&= x_n - y_{n-1} - R_\alpha(x_n)^{-1}[A_0^*(F(x_n) - F(y_{n-1}) + \alpha(x_n - y_{n-1})] \\
&\quad + [R_\alpha(x_{n-1})^{-1} - R_\alpha(x_n)^{-1}][A_0^*(F(y_{n-1}) - f^\delta) + \alpha(y_{n-1} - x_0)] \\
&= R_\alpha(x_n)^{-1}[R_\alpha(x_n)(x_n - y_{n-1}) - A_0^*(F(x_n) - F(y_{n-1})) \\
&\quad - \alpha(x_n - y_{n-1})] + [R_\alpha(x_{n-1})^{-1} - R_\alpha(x_n)^{-1}] \\
&\quad \times [A_0^*(F(y_{n-1}) - f^\delta) + \alpha(y_{n-1} - x_0)]. \qquad (2.9)
\end{aligned}
$$

Then, again by Assumption 1.1 and (2.9) we obtain that

$$
\begin{aligned}
e_n &\leq \|R_\alpha(x_n)^{-1} A_0^* \int_0^1 [F'(x_n) - F'(y_{n-1} + t(x_n - y_{n-1}))]dt(x_n - y_{n-1})\| \\
&\quad + \|R_\alpha(x_n)^{-1}(F'(x_n) - F'(x_{n-1}))R_\alpha(x_{n-1})^{-1}[A_0^*(F(y_{n-1}) - f^\delta) \\
&\quad + \alpha(y_{n-1} - x_0)]\| \\
&\leq \|R_\alpha(x_n)^{-1} A_0^* \int_0^1 [F'(x_n) - F'(y_{n-1} + t(x_n - y_{n-1}))]dt(x_n - y_{n-1})\| \\
&\quad + \|R_\alpha(x_n)^{-1}(F'(x_n) - F'(x_{n-1}))(y_{n-1} - x_n)\| \\
&\leq \frac{1}{1 - L_0 r}[L_0[\|x_n - x_0\| + \int_0^1 \|y_{n-1} - x_0 + t(x_n - y_{n-1})\|dt]\|x_n - y_{n-1}\|] \\
&\quad + L_0[\|x_n - x_0\| + \|x_{n-1} - x_0\|]\|x_n - y_{n-1}\| \\
&\leq \frac{1}{1 - L_0 r}[4L_0 r\|y_{n-1} - x_n\|] = 4L_0 r(2L_0 r)e_{n-1} \\
&= q e_{n-1}.
\end{aligned}
$$

This completes the proof of the Theorem.

THEOREM 2.4. *Under the hypotheses of Theorem 2.3 further suppose that*

$$\rho < \rho_0 \text{ and } L_0 \leq 1. \qquad (2.10)$$

Moreover, suppose that

$$\overline{U(x_0, r)} \subseteq D(F). \qquad (2.11)$$

Then, $x_n, y_n \in U(x_0, r)$ *for each* $n = 0, 1, 2, \cdots$.

Proof. We note by (2.10) that we have

$$q \in (0, 1). \qquad (2.12)$$

Using Lemma 2.2, Theorem 2.3 and (2.11) we get that

$$\|x_1 - x_0\| \leq (1 + L_0 r)e_0 \leq (1 + L_0 r)b_\rho < r.$$

Hence, $x_1 \in U(x_0, r)$. Similarly, we obtain that

$$\begin{aligned}
\|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - x_0\| && (2.13)\\
&\leq qe_0 + (1 + \frac{p}{2})b_\rho && (2.14)\\
&\leq [q + 1 + \frac{p}{2}]b_\rho < L_0 r \leq r, && (2.15)
\end{aligned}$$

which implies $y_1 \in U(x_0, r)$. Moreover, we have that

$$\begin{aligned}
\|x_2 - x_0\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\|\\
&\leq (1 + \frac{p}{2})\|y_1 - x_1\| + (1 + \frac{p}{2})b_\rho\\
&\leq (1 + \frac{p}{2})qb_\rho + (1 + \frac{p}{2})b_\rho\\
&= (1 + q)(1 + \frac{p}{2})b_\rho < L_0 r \leq r,
\end{aligned}$$

which also implies $x_2 \in U(x_0, r)$. Furthermore, we obtain that

$$\begin{aligned}
\|y_2 - x_0\| &\leq \|y_2 - x_2\| + \|x_2 - x_0\|\\
&\leq q\|y_1 - x_1\| + (1 + q)(1 + \frac{p}{2})b_\rho\\
&\leq q^2(1 + \frac{p}{2})b_\rho + (1 + q)(1 + \frac{p}{2})b_\rho\\
&\leq (1 + q + q^2)(1 + \frac{p}{2})b_\rho < L_0 r \leq r.
\end{aligned}$$

Hence, we proved that $y_2 \in U(x_0, r)$. Proceeding in an analogous way we prove that $x_n, y_n \in U(x_0, r)$. That completes the proof of the Theorem.

THEOREM 2.5. *Suppose that the hypotheses of Theorem 2.4 hold. Then, sequence $\{x^\delta_{n,\alpha}\}$ remains in $U(x_0, r)$ for each $n = 0, 1, 2, \cdots$ and converges to a solution $x^\delta_\alpha \in \overline{U(x_0, r)}$ of equation (1.2). Moreover, the following estimates hold*

$$\|x_n - x^\delta_\alpha\| \leq b_0 e^{-\gamma_0 n}, \tag{2.16}$$

where $b_0 = (1 + \frac{p}{2})\gamma_\rho$ and $\gamma_0 = -\ln q > 0$.

Proof. Using (b) of Theorem 2.3 and (2.10) we get that

$$\|x_{n+m} - x_n\| \leq \sum_{i=0}^{m-1} \|x_{n+i+1} - x_{n+i}\|. \tag{2.17}$$

But, we have

$$\|x_{n+i+1} - x_{n+i}\| \le (1+\frac{p}{2})q^{n+i}e_0. \tag{2.18}$$

In view of (2.18), inequality (2.17) gives that

$$\begin{aligned} \|x_{n+m} - x_n\| &\le [1 + q + q^2 + \cdots + q^{m-1}]q^n(1+\frac{p}{2})e_0 \\ &\le \frac{1-q^m}{1-q}(1+\frac{p}{2})q^n e_0. \end{aligned} \tag{2.19}$$

It follows from (2.19) that sequence $\{x_n\}$ is complete in a Hilbert space X and as such it converges to some $x_\alpha^\delta \in \overline{U(x_0, r)}$ (since $\overline{U(x_0, r)}$ is closed set). By letting $m \to \infty$ we obtain (2.16). Finally, to prove x_α^δ is a solution of (1.2), note that

$$\begin{aligned} \|A_0^*(F(x_n) - f^\delta) + \alpha(x_n - x_0)\| &= \|R_\alpha(x_n)(x_n - y_n)\| \\ &\le (\|A_0^* F'(x_n)\| + \alpha)e_n \\ &\le (\|A_0^* F'(x_n)\| + \alpha)q^n\gamma_\rho \to 0 \text{ as } n \to \infty. \end{aligned}$$

That completes the proof of the Theorem.

REMARK 2.6. (a) *The convergence order of (TSNTM) is three* [7] *under* $(C1)$. *In Theorem 2.5 the error bounds are too pessimistic. That is why in practice we shall use the computational order of convergence (COC) (see eg.* [5]*) defined by*

$$\varrho \approx \ln\left(\frac{\|x_{n+1} - x_\alpha^\delta\|}{\|x_n - x_\alpha^\delta\|}\right) / \ln\left(\frac{\|x_n - x_\alpha^\delta\|}{\|x_{n-1} - x_\alpha^\delta\|}\right).$$

(b) *In the rest of this section we suppose that*

$$\rho_0 \le r \tag{2.20}$$

which is possible for x_0 sufficiently close to $\hat{x}$.

3. Error analysis

Next, we present the results concerning error bounds under source conditions. We need a condition on the source function.

ASSUMPTION 3.1. *There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \ge \|A_0\|^2$ satisfying $lim_{\lambda \to 0}\varphi(\lambda) = 0$ and $v \in X$ with $\|v\| \le 1$ such that*

$$x_0 - \hat{x} = \varphi(A_0^* A_0)v$$

and

$$sup_{\lambda \ge 0}\frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \le c_\varphi \varphi(\alpha), \quad \forall \lambda \in (0, a].$$

REMARK 3.2. *It can easily be seen that functions*

$$\varphi(\lambda) = \lambda^\nu, \quad \lambda > 0$$

for $0 < \nu \leq 1$ *and*

$$\varphi(\lambda) = \begin{cases} (ln\frac{1}{\lambda})^{-\beta} & , \quad 0 < \lambda \leq e^{-(\beta+1)} \\ 0 & , \quad otherwise \end{cases}$$

for $\beta \geq 0$ *satisfy* $(C2)$ *(cf.* [35]*).*

THEOREM 3.3. [30, Theorem 3.1] *Let* x_α^δ *be as in (1.8),* r *be as in (2.4) and let* $q = L_0 r$. *Suppose Assumptions 1.1 and Assumption 3.1 hold. Then*

$$\|x_\alpha^\delta - \hat{x}\| \leq \frac{1}{1-q}(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)).$$

THEOREM 3.4. *Suppose hypotheses of Theorem 2.5 and Theorem 3.3 hold. Then, the following assertion holds*

$$\|x_n - \hat{x}\| \leq b_0 e^{-\gamma_0 n} + \frac{1}{1-q}(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)).$$

Let

$$n_\delta := min\{n : e^{-\gamma_0 n} \leq \frac{\delta}{\sqrt{\alpha}}\}. \tag{3.1}$$

THEOREM 3.5. *Let* n_δ *be as in (3.1). Suppose that hypothese of Theorem 3.4 hold. Then, the following assertions hold*

$$\|x_{n_\delta} - \hat{x}\| \leq \frac{1+b_0}{1-q}(\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}). \tag{3.2}$$

Note that the error estimate $\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}$ in (3.2) is of optimal order if $\alpha := \alpha_\delta$ satisfies, $\varphi(\alpha_\delta)\sqrt{\alpha_\delta} = \delta$.

Now using the function $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq a$ we have $\delta = \sqrt{\alpha_\delta}\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. In view of the above observations and (3.2) we have the following.

THEOREM 3.6. *Let* $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}$ *for* $0 < \lambda \leq a$, *and the assumptions in Theorem 3.5 hold. For* $\delta > 0$, *let* $\alpha := \alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$ *and let* n_δ *be as in (3.1). Then*

$$\|x_{n_\delta} - \hat{x}\| = O(\psi^{-1}(\delta)).$$

In this section, we present a parameter choice rule based on the balancing principle studied in [21]. In this method, the regularization parameter α is selected from some finite set

$$D_M(\alpha) := \{\alpha_i = \mu^i \alpha_0, i = 0, 1, \cdots, M\}$$

where $\mu > 1$, $\alpha_0 > 0$ and let

$$n_i := min\{n : e^{-\gamma_0 n} \leq \frac{\delta}{\sqrt{\alpha_i}}\}.$$

Then for $i = 0, 1, \cdots, M$, we have

$$\|x^\delta_{n_i,\alpha_i} - x^\delta_{\alpha_i}\| \leq c\frac{\delta}{\sqrt{\alpha_i}}, \quad \forall i = 0, 1, \cdots M.$$

Let $x_i := x^\delta_{n_i,\alpha_i}$. The parameter choice strategy that we are going to consider in this paper, we select $\alpha = \alpha_i$ from $D_M(\alpha)$ and operate only with corresponding $x_i, \quad i = 0, 1, \cdots, M$. Proof of the following theorem is analogous to the proof of Theorem 4.4 in [15] (see also [16]).

THEOREM 3.7. *(cf.* [15]*, Theorem 4.4) Assume that there exists* $i \in \{0, 1, 2, \cdots, M\}$ *such that* $\varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}$*. Suppose the hypotheses of Theorem 3.5 and Theorem 3.6 hold and let*

$$l := max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}\} < M,$$

$$k := max\{i : \|x_i - x_j\| \leq 4\bar{c}\frac{\delta}{\sqrt{\alpha_j}}, \quad j = 0, 1, 2, \cdots, i\}.$$

Then $l \leq k$ *and*

$$\|\hat{x} - x_k\| \leq c\psi^{-1}(\delta)$$

where $c = 6\bar{c}\mu$*.*

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 3.7 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta_0 < (17 - 12\sqrt{2})\sqrt{\alpha_0}$ and $\mu > 1$.
- Choose M big enough but not too large and $\alpha_i := \mu^i\alpha_0, i = 0, 1, 2, \cdots, M$.
- Choose $\rho \leq \rho_0$.

3.1. **Algorithm.**

1. Set $i = 0$.
2. Choose $n_i = min\{n : e^{-\gamma_0 n} \leq \frac{\delta}{\sqrt{\alpha_i}}\}$.
3. Solve $x_i = x^\delta_{n_i,\alpha_i}$ by using the iteration (1.8).
4. If $\|x_i - x_j\| > 4\bar{c}\frac{\delta}{\sqrt{\alpha_j}}, j < i$, then take $k = i - 1$ and return x_k.
5. Else set $i = i + 1$ and return to Step 2.

4. Examples

Next we present two examples where $(C1)$ is not satisfied but Assumption 1.1 is satisfied.

EXAMPLE 4.1. *Let $X = Y = \mathbb{R}$, $D = [0, \infty)$, $x_0 = 1$ and define function F on D by*

$$F(x) = \frac{x^{1+\frac{1}{i}}}{1+\frac{1}{i}} + c_1 x + c_2, \tag{4.1}$$

where c_1, c_2 are real parameters and $i > 2$ an integer. Then $F'(x) = x^{1/i} + c_1$ is not Lipschitz on D. That is (C1) cannot be satisfied. However, Assumption 1.1 holds for $L_0 = 1$.

Indeed, we have

$$\begin{aligned} \|F'(x) - F'(x_0)\| &= |x^{1/i} - x_0^{1/i}| \\ &= \frac{|x - x_0|}{x_0^{\frac{i-1}{i}} + \cdots + x^{\frac{i-1}{i}}} \\ &\leq L_0 |x - x_0|. \end{aligned}$$

EXAMPLE 4.2. *We consider the integral equations*

$$u(s) = f(s) + \tau \int_a^b G(s,t) u(t)^{1+1/n} dt, \quad n \in \mathbb{N}. \tag{4.2}$$

Here, f is a given continuous function satifying $f(s) > 0, s \in [a,b], \tau$ is a real number, and the kernel G is continuous and positive in $[a,b] \times [a,b]$.

For example, when $G(s,t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$u'' = \tau u^{1+1/n} \tag{4.3}$$

$$u(a) = f(a), u(b) = f(b). \tag{4.4}$$

These type of problems have been considered in **[?]**, [2], [34].

Equation of the form (4.2) generalize equations of the form

$$u(s) = \int_a^b G(s,t) u(t)^n dt \tag{4.5}$$

studied in **[?]**, [2], [34]. *Instead of (4.2) we can try to solve the equation $F(u) = 0$ where*

$$F : \Omega \subseteq C[a,b] \to C[a,b], \Omega = \{u \in C[a,b] : u(s) \geq 0, s \in [a,b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \tau \int_a^b G(s,t) u(t)^{1+1/n} dt.$$

The norm we consider is the max-norm.

The derivative F' is given by

$$F'(u)v(s) = v(s) - \tau(1+\frac{1}{n})\int_a^b G(s,t)u(t)^{1/n}v(t)dt, \ \ v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b]=[0,1], G(s,t)=1$ and $y(t)=0$. Then $F'(y)v(s)=v(s)$ and

$$\|F'(x)-F'(y)\| = |\tau|(1+\frac{1}{n})\int_a^b x(t)^{1/n}dt.$$

If F' were a Lipschitz function, then

$$\|F'(x)-F'(y)\| \leq L_1\|x-y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n}dt \leq L_2 \max_{x\in[0,1]} x(s), \tag{4.6}$$

would hold for all $x \in \Omega$ and for a constant L_2. But this is not true. Consider, for example, the functions

$$x_j(t) = \frac{t}{j}, \ \ j \geq 1, \ \ t \in [0,1].$$

If these are substituted into (4.6)

$$\frac{1}{j^{1/n}(1+1/n)} \leq \frac{L_2}{j} \Leftrightarrow j^{1-1/n} \leq L_2(1+1/n), \ \ \forall j \geq 1.$$

This inequality is not true when $j \to \infty$.

Therefore, condition (4.6) is not satisfied in this case. However, Assumption 1.1 holds. To show this, let $x_0(t)=f(t)$ and $\gamma = \min_{s\in[a,b]} f(s), \alpha > 0$ Then for $v \in \Omega$,

$$\begin{aligned}\|[F'(x)-F'(x_0)]v\| &= |\tau|(1+\frac{1}{n})\max_{s\in[a,b]}|\int_a^b G(s,t)(x(t)^{1/n}-f(t)^{1/n})v(t)dt| \\ &\leq |\tau|(1+\frac{1}{n})\max_{s\in[a,b]} G_n(s,t)\end{aligned}$$

where $G_n(s,t) = \frac{G(s,t)|x(t)-f(t)|}{x(t)^{(n-1)/n}+x(t)^{(n-2)/n}f(t)^{1/n}+\cdots+f(t)^{(n-1)/n}}\|v\|.$

Hence,

$$\begin{aligned}\|[F'(x)-F'(x_0)]v\| &= \frac{|\tau|(1+1/n)}{\gamma^{(n-1)/n}}\max_{s\in[a,b]}\int_a^b G(s,t)dt\|x-x_0\| \\ &\leq L_0\|x-x_0\|,\end{aligned}$$

where $L_0 = \frac{|\tau|(1+1/n)}{\gamma^{(n-1)/n}}N$ and $N = \max_{s\in[a,b]}\int_a^b G(s,t)dt$. Then Assumption 1.1 holds for sufficiently small τ. Other examples where $L_0 < L$ or L does not exist can be found in [1, 5].

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