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APPROXIMATION OF A FUZZY NUMBER BY TWO MAIN CHARACTERISTICS

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ABSTRACT. In this paper, we introduce a trapezoidal approximation of an arbitrary fuzzy number by two main characteristics. Core and support of the fuzzy number are considered as two important characteristics of the fuzzy number. The proposed approximation will preserve mean-core and mean-support of the fuzzy number. The operator so called trapezoidal approximation Pmcs. In case that the mean-core and mean-support are identical, the trapezoidal approximation Pmcs is symmetric. We then discuss properties of the approximation strategy including translation invariance, scale invariance and identity. The advantage is that two important qualifications of fuzzy number will be considered. Moreover, the proposed method will be simple computationally and natural. The method is illustrated by some numerical examples.

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1. INTRODUCTION

Fuzzy numbers appear as the most popular family of fuzzy sets useful both for theoretical consideration as well as diverse practical applications. However, complicated membership functions have many drawbacks in processing imprecise information modeled by fuzzy numbers including problems with calculations, computer implementation, etc. Moreover, handling too complex membership functions entails difficulties in interpretation of the results too. This is the reason that a suitable approximation of fuzzy numbers is so important. Less regular membership functions lead to calculations that are more complicated. A natural need is to approximate fuzzy numbers with the simpler shapes which are easy to handle and have natural interpretations. The most commonly used method is to approximate it with a crisp real number, which is also called defuzzification. So far many defuzzification methods are proposed

[1, 2, 3, 4, 5, 6]. For the sake of simplicity, the trapezoidal or triangular fuzzy numbers are most common in current applications. The importance of the approximation of fuzzy numbers by trapezoidal fuzzy numbers is pointed out in many papers [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. The symmetric triangular approximation was presented by Ma et al. [17], Chanas [18] derived a formula for determining the interval approximations under the hamming distance. Other approximations were proposed by Abbasbandy et al. [8, 9, 10]. These works show that the approximation and ordering of fuzzy numbers are meaningful topics.

The reminder of this paper is organized as follows: Section 2 contains some basic notation of fuzzy numbers. In Section 3, we investigate the trapezoidal approximation of an arbitrary fuzzy number based on the mean-core (Mc) and the mean-support (Ms) then the method will be illustrated by two numerical examples. In addition, we discuss some properties of trapezoidal approximation Pmcs such as translation invariance, scale invariance, identity. Concluding remarks are given in Section 4.

2. Preliminaries

However, there are a number of ways of defining fuzzy numbers, for the purposes of this paper we adopt the following definition; we will identify the name of the number with its membership function for simplicity. Throughout this paper, $\mathbb{R}$ stands for the set of all real numbers, $F(\mathbb{R})$ stands the set of fuzzy numbers, A expresses a fuzzy number and $A(x)$ for its membership function, $\forall x \in \mathbb{R}$.

Definition 2.1. [19, 20] *A fuzzy subset A of the real line $\mathbb{R}$ with membership function $A(x),\ A : \mathbb{R} \to [0, 1]$, is called a fuzzy number if*

(a) A is normal, i.e., there exist an element x_0 such that $A(x_0) = 1$,
(b) A is fuzzy convex, i.e., $A(\lambda x_1 + (1 - \lambda)x_2) \geq A(x_1) \wedge A(x_2)$,
(c) $A(x)$ is upper semi-continuous,
(d) $supp(A)$ is bounded, where $supp(A) = cl\{x \in \mathbb{R} : A(x) > 0\}$, and *cl* is the closure operator.

It is known that for fuzzy number A there exist four numbers $a, b, c, d \in \mathbb{R}$ and two functions $L_A(x),\ R_A(x) : \mathbb{R} \to [0, 1]$, where $L_A(x)$ and $R_A(x)$ are non-decreasing and non-increasing functions, respectively. We can describe a membership function as follows:

$$A(x) = \begin{cases} 0 & x \leq a, \\ L_A(x) & a \leq x \leq b, \\ 1 & b \leq x \leq c, \\ R_A(x) & c \leq x \leq d, \\ 0 & d < x. \end{cases}$$

The functions $L_A(x)$ and $R_A(x)$ are also called the left and right side of the fuzzy number A, respectively [19, 20].

In this paper, we assume that

$$\int_{-\infty}^{+\infty} A(x)dx < +\infty.$$

A useful tool for dealing with fuzzy numbers are their $\alpha-$cuts. The $\alpha-$cut of a fuzzy number A is non-fuzzy set defined as

$$A_\alpha = \{x \in \mathbb{R} : A(x) \geq \alpha\},$$

for $\alpha \in (0, 1]$ and $A_0 = cl(\cup_{\alpha\in(0,1]} A_\alpha)$. According to the definition of a fuzzy number, it is seen at once that every $\alpha-$cut of a fuzzy number is closed interval. Hence, for a fuzzy number A, we have $A(\alpha) = [A_L(\alpha), A_R(\alpha)]$ where

$$A_L(\alpha) = \inf\{x \in \mathbb{R} : A(x) \geq \alpha\},$$

$$A_R(\alpha) = \sup\{x \in \mathbb{R} : A(x) \geq \alpha\}.$$

If the left and right sides of the fuzzy number A are strictly monotone, obviously, A_L and A_R are inverse functions of $L_A(x)$ and $R_A(x)$, respectively. Another important kind of fuzzy numbers was introduced in [21] as follows: Let $a, b, c, d \in \mathbb{R}$ such that $a < b \leq c < d$. A fuzzy number A defined as $A(x),\ A : \mathbb{R} \to [0, 1]$,

$$A(x) = \begin{cases} 0 & x \leq a, \\ (\frac{x-a}{b-a})^r & a \leq x \leq b \\ 1 & b \leq x \leq c, \\ (\frac{d-x}{d-c})^r & c < x \leq d, \\ 0 & d < x, \end{cases}$$

where $r > 0,$ is denoted by $A = (a, b, c, d)_r$. If $A = (a, b, c, d)_r$ then

$$A_\alpha = \big[A_L(\alpha), A_R(\alpha)\big] = \big[a + (b - a)\alpha^{1/r},\ d - (d - c)\alpha^{1/r}\big],\ \alpha \in [0, 1].$$

When $r = 1$ and $b = c$ we obtain, so called, triangular fuzzy number, the conditions $r = 1,\ a = b$ and $c = d$ imply the close interval and in case $r = 1,\ a = b = c = d = t$ we have the crisp number t.

Since the trapezoidal fuzzy number is completely characterized by $r = 1$ and four real numbers $a \leq b \leq c \leq d$ it is often denoted in brief as $A = (a, b, c, d)$. A family of trapezoidal fuzzy number will be denoted by $F^T(\mathbb{R})$.

As we mentioned for two arbitrary fuzzy numbers A and B with $\alpha-$cut sets $[A_L(\alpha), A_R(\alpha)]$ and $[B_L(\alpha), B_R(\alpha)]$, respectively, the equality

$$d(A,B) = \sqrt{\int_0^1 \left(A_L(\alpha) - B_L(\alpha)\right)^2 \mathrm{d}\alpha + \int_0^1 \left(A_R(\alpha) - B_R(\alpha)\right)^2 \mathrm{d}\alpha}, \tag{1}$$

is the distance between A and B. For more details we refer the reader to [22]. The set of all elements that have a nonzero degree of membership in $\tilde{A}$ is called the *support* of A, i.e.

$$supp(A) = \{x \in X \mid f_{\tilde{A}}(x) > 0\}. \tag{2}$$

The set of elements having the largest degree of membership in A is called the *core* of A, i.e.

$$core(A) = \{x \in X \mid f_{\tilde{A}}(x) = \sup_{x \in X} f_{\tilde{A}}(x)\}. \tag{3}$$

In the following, we will always assume that A is continuous and bounded support $supp(A) = (a, d)$. The strong support of A should be $\overline{supp}\,(A) = [a, d]$.

2.1. **Mean-Core (Mc).** The Mc takes the average of the first and the last values from core(A), i.e.

$$Mc(A) = \frac{A_L(1) + A_R(1)}{2}. \tag{4}$$

In the case that $A = (a, b, c, d)$ be a trapezoidal fuzzy number $Mc(A) = \frac{b+c}{2}$.

2.2. **Mean-Support (Ms).** The Ms takes the average of the first and the last values from support(A), i.e.

$$Ms(A) = \frac{A_L(0) + A_R(0)}{2}. \tag{5}$$

In the case that $A = (a, b, c, d)$ be a trapezoidal fuzzy number $Ms(A) = \frac{a+d}{2}$.

3. Trapezoidal Approximation of a Fuzzy Number Based on Mc-Ms

In this section, we will propose the nearest trapesoidal approximation of an arbitrary fuzzy number, which preserves Mc and Ms of the fuzzy number, then we call it trapezoidal approximation Pmcs.

Suppose A is a fuzzy number with $\alpha-$cut sets $\big(A_L(\alpha), A_R(\alpha)\big)$. Given A, we will try to find a trapezoidal fuzzy number $T(A)$ as follows:

$$T(A) = \Big(s_A - |\beta_{T(A)}|,\ c_A - |\sigma_{T(A)}|,\ c_A + |\sigma_{T(A)}|,\ s_A + |\beta_{T(A)}|\Big),$$

$$s_A - |\beta_{T(A)}| \leq c_A - |\sigma_{T(A)}| \leq c_A + |\sigma_{T(A)}| \leq s_A + |\beta_{T(A)}|,$$

which is the nearest to A with respect to metric D, such that

$$c_A = Moc(\tilde{A}),\ s_A = Mos(\tilde{A})$$

and $\sigma_{T(A)}, \beta_{T(A)}$ are real numbers. Let $\big(T_L(\alpha), T_R(\alpha)\big)$ denotes α-cut sets of $T(A)$, we would like to minimize

$$d_f(A, T(A)) = \sqrt{\int_0^1 \big(A_L(\alpha) - T_L(A)(\alpha)\big)^2 \mathbf{d}\alpha + \int_0^1 \big(A_R(\alpha) - T_R(A)(\alpha)\big)^2 \mathbf{d}\alpha}, \tag{6}$$

Therefore, (6) reduces to

$$\begin{aligned} d_f(A, T(A)) = \Big[\int_0^1 \Big(s_{T(A)} - |\beta_{T(A)}| + \big(c_A - |\sigma_{T(A)}| - s_A + |\beta_{T(A)}|\big)\alpha - A_L(\alpha)\Big)^2 \mathbf{d}\alpha \\ + \int_0^1 \Big(s_A + |\beta_{T(A)}| + \big(c_A + |\sigma_{T(A)}| - s_{T(A)} - |\beta_{T(A)}|\big)\alpha - A_R(\alpha)\Big)^2 \mathbf{d}\alpha\Big]^{\frac{1}{2}}, \end{aligned} \tag{7}$$

and we will try to minimize (7) with respect to $\sigma_{T(A)}$ and $\beta_{T(A)}$. To emphasize, we want to find a trapezoidal fuzzy number, which is not only closest to given fuzzy number but which preserves Mc and Ms of the fuzzy number.
It is easily seen that in order to minimize $d_f(A, T(A))$ it suffices function

$$d_f^2(A, T(A)) = D(\sigma_{T(A)}, \beta_{T(A)}).$$

Consequently, we can get their partial derivatives

$$\frac{\partial D\big(\sigma_{T(A)}, \beta_{T(A)}\big)}{\partial \sigma_{T(A)}} =$$

$$2\int_0^1 \Big[2|\beta_{T(A)}|(1-\alpha)\alpha + 2|\sigma_{T(A)}|\alpha^2 - \big(A_R(\alpha) - A_L(\alpha)\big)\alpha\Big] \frac{\sigma_{T(A)}}{|\sigma_{T(A)}|}\mathbf{d}\alpha, \tag{8}$$

$$\frac{\partial D\big(\sigma_{T(A)}, \beta_{T(A)}\big)}{\partial \beta_{T(A)}} =$$

$$2\int_0^1 \Big[2|\beta_{T(A)}|(1-\alpha)^2 + 2|\sigma_{T(A)}|(1-\alpha)\alpha - \big(A_R(\alpha) - A_L(\alpha)\big)(1-\alpha)\Big] \frac{\beta_{T(A)}}{|\beta_{T(A)}|}\mathbf{d}\alpha. \tag{9}$$

Let

$$\frac{\partial D(\sigma_{T(A)}, \beta_{T(A)})}{\partial |\sigma_{T(A)}|} = \frac{\partial D(\sigma_{T(A)}, \beta_{T(A)})}{\partial |\beta_{T(A)}|} = 0. \tag{10}$$

by replacing

$$\int_0^1 (1-\alpha)\alpha \mathbf{d}\alpha = \frac{1}{6}, \int_0^1 (1-\alpha)^2 = \frac{1}{3}, \int_0^1 \alpha \mathbf{d}\alpha = \frac{1}{2},$$

we can get that the following system

$$\begin{cases} \frac{2}{3}|\sigma_{T(A)}| + \frac{1}{3}|\beta_{T(A)}| = \int_0^1 \big(A_R(\alpha) - A_L(\alpha)\big)\alpha \mathbf{d}\alpha \\ \\ \frac{1}{3}|\sigma_{T(A)}| + \frac{2}{3}|\beta_{T(A)}| = \int_0^1 \big(A_R(\alpha) - A_L(\alpha)\big)(1-\alpha)\mathbf{d}\alpha. \end{cases} \tag{11}$$

Therefore, we have four systems. In short the solution is

$$\begin{cases} \sigma_{T(A)} = \big| \int_0^1 \big(A_R(\alpha) - A_L(\alpha)\big)(3\alpha - 1)\mathbf{d}\alpha \big| \\ \\ \beta_{T(A)} = \big| \int_0^1 \big(A_R(\alpha) - A_L(\alpha)\big)(2 - 3\alpha)\mathbf{d}\alpha \big|. \end{cases} \tag{12}$$

Remark 3.1. *Let A be a trapezoidal fuzzy number. In case that Mc and Ms are identical, the trapezoidal approximation Pmcs is a symmetric trapezoidal fuzzy number.*

Example 3.1. [10] *Let A be a fuzzy number with the following membership function (Fig.1)*

$$\tilde{A}(x) = \begin{cases} 1 - \frac{(x-5)^2}{4} & 3 \leq x \leq 7, \\ 0 & otherwise, \end{cases}$$

Its $\alpha-$cut representation is

$$A_L(\alpha) = 5 - 2\sqrt{1-\alpha},\ A_R(\alpha) = 5 + 2\sqrt{1-\alpha},$$

from relation (12) we can get that

$$c_A = 5,\ s_A = 5,\ \sigma_{T(A)} = \frac{8}{15},\ \beta_{T(A)} = \frac{32}{15},$$

therefore,

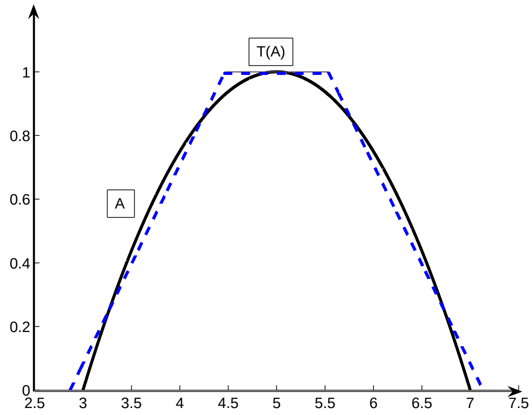

FIGURE 1

$$T(A) = (\frac{43}{15}, \frac{67}{15}, \frac{83}{15}, \frac{107}{15}).$$

Since c_A and s_A are equal, then the approximation of fuzzy number A is a symmetric trapezoidal fuzzy number. Moreover, in this case the result is identical to Abbasbandy and Asady's work [10].

Example 3.2. [3, 11] *Let us considering the fuzzy number A with membership function*

$$\tilde{A}(x) = \begin{cases} \frac{x}{2}+1 & -2 \leq x \leq 0, \\ (x-1)^2 & 0 \leq x \leq 1, \\ 0 & otherwise, \end{cases}$$

which is indicated in Fig.2. Its $\alpha-$cut representation is

$$A_L(\alpha) = 2\alpha - 2, \; A_R(\alpha) = 1 - \sqrt{\alpha},$$

from relation (12) we can get that

$$c_A = 0, \; s_A = \frac{-1}{2}, \; \sigma_{T(A)} = \frac{1}{30}, \; \beta_{T(A)} = \frac{41}{30},$$

therefore,

$$T(A) = (\frac{-28}{15}, \frac{-1}{30}, \frac{1}{30}, \frac{13}{15}).$$

By Ban's method the trapezoidal approximation is

$$T(A)_B = (\frac{-59}{30}, \frac{-1}{30}, \frac{-1}{30}, \frac{7}{10})$$

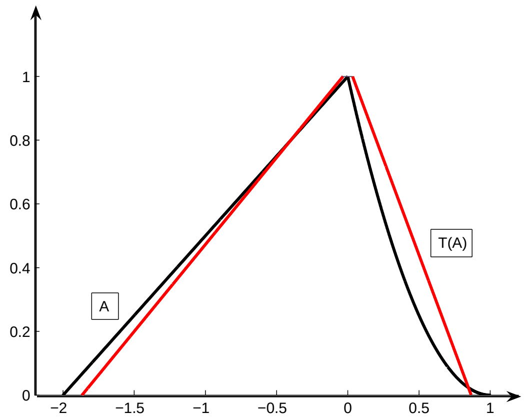

FIGURE 2

Theorem 3.1. *The trapezoidal approximation Pmcs is invariant to translation.*

Proof. Let z be a real number and A denotes a fuzzy number with $\alpha-$cut $A_\alpha = \big[A_L(\alpha), A_R(\alpha)\big]$. Then the $\alpha-$cut of a fuzzy number A translated by a number z is

$$(A+z)_\alpha = \big[A_L(\alpha)+z, A_R(\alpha)+z\big].$$

Now consider

$$T(A) = (s_A - \beta_{T(A)},\ c_A - \sigma_{T(A)},\ c_A + \sigma_{T(A)},\ s_A + \beta_{T(A)}).$$

Accordingly, from Eqs. (12) we can get that

$$\sigma_{A+z} = \sigma_{T(A)},\ \beta_{A+z} = \beta_{T(A)}.$$

Moreover,

$$s_{A+z} = s_A + z,\ c_{A+z} = c_A + z.$$

Consequently,

$$T(A+z) = (s_A + z - \beta_{T(A)}, c_A + z - \sigma_{T(A)}, c_A + z + \sigma_{T(A)}, s_A + z + \beta_{T(A)}),$$

it shows that $T(A+z) = T(A) + z$, which proves the translation invariance.

Theorem 3.2. *The trapezoidal approximation based Pmcs is invariant to scale.*

Proof. Consider a real number λ such that $\lambda \neq 0$. Using Eqs. (12) we can get that

$$\sigma_{T(\lambda A)} = \lambda\sigma_{T(A)},\ \beta_{T(\lambda A)} = \lambda\beta_{T(A)},\ c_{\lambda A} = \lambda c_A,\ s_{\lambda A} = \lambda s_A.$$

Moreover,

$$T(\lambda A) = (\lambda s_A - \lambda\beta_{T(A)}, \lambda c_A - \lambda\sigma_{T(A)}, \lambda c_A + \lambda\sigma_{T(A)}, \lambda s_A + \lambda\beta_{T(A)}),$$

then we have $T(\lambda A) = \lambda T(A)$, which proves the scale invariance.

Theorem 3.3. *The weighted trapezoidal approximation operator-preserving core satisfies the identity property.*

Proof. Suppose a trapezoidal fuzzy number $A = (a, b, c, d)$ where $a \leq b \leq c \leq d$ and its $\alpha-$cut set $A(\alpha) = \big[a + (b-a)\alpha, d - (d-c)\alpha\big]$.

Applying Eqs. (12) we can get that

$$\sigma_{T(A)} = \int_0^1 \big[d - a + (a - b + c - d)\alpha\big](3\alpha - 1),$$

$$\beta_{T(A)} = \int_0^1 \big[d - a + (a - b + c - d)\alpha\big](2\alpha - 3).$$

Hence,

$$\sigma_{T(A)} = \frac{c-b}{2},\ \beta_{T(A)} = \frac{d-a}{2}.$$

Since,

$$c_A = \frac{b+c}{2},\ s_A = \frac{a+d}{2},$$

consequently,

$$c_A - \sigma = b,\ c_A + \sigma_{T(A)} = c$$

and

$$s_A - \beta_{T(A)} = a,\ s_A + \beta_{T(A)} = d.$$

It means that $T(A) = (a, b, c, d)$, or $T(A) = A$, which proved that the trapezoidal approximation Pmcs satisfies the identity property.

4. Conclusions

In the present contribution, we used ordinary distance between two fuzzy numbers to investigate a trapezoidal approximation of arbitrary fuzzy numbers. In this method, the middle of core and the middle of support of the fuzzy number are preserved. The proposed operator so called trapezoidal approximation Pmcs. A satisfactory approximation operator should be easy to implement, computational and inexpensive. We also discussed some properties of the approximation including translation invariance, scale invariance and identity. The advantage is that two important qualifications of fuzzy number are considered. In addition, the proposed method is simple computationally and natural.

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