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INVERSE LAPLACE TRANSFORM FOR BI-COMPLEX VARIABLES

A. BANERJEE[1], S. K. DATTA[2] AND MD. A. HOQUE[3]

[1]Department of Mathematics, Krishnath College, Berhampore,
Murshidabad 742101, India
[2]Department of Mathematics, University of Kalyani, Kalyani, Nadia,
PIN-741235,India
[3]Gobargara High Madrasah (H. S.), Hariharpara, Murshidabad
742166, India

Abstract. In this paper we examine the existence of bicomplexified inverse Laplace transform as an extension of its complexified inverse version within the region of convergence of bicomplex Laplace transform. In this course we use the idempotent representation of bicomplex-valued functions as projections on the auxiliary complex spaces of the components of bicomplex numbers along two orthogonal,idempotent hyperbolic directions.

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1. Introduction

The theory of bicomplex numbers is a matter of active research for quite a long time science the seminal work of Segre[1] in search of special algebra.The algebra of bicomplex numbers are widely used in the literature as it becomes a valiable commutative alternative [2] to the non-commutative skew field of quaternions introduced by Hamilton [3] (both are four- dimensional and generalization of complex numbers).

A bicomplex number is defined as

$$\xi = a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3,$$

where $a_{0,}\ a_1, a_2, a_3$ are real numbers, $i_1^2 = i_2^2 = -1$ and

$$i_1i_2 = i_2i_1, (i_1i_2)^2 = 1.$$

The set of bicomplex numbers,complex numbers and real numbers are denoted by C_2, C_1,and C_0 respectively. C_2 becomes a Real Commutative Algebra with identity

$$1 = 1 + i_1 \cdot 0 + i_2 \cdot 0 + i_1i_2 \cdot 0$$

with standard binary composition.

There are two non trivial elements $e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$ in C_2 with the properties $e_1^2 = e_1, e_2^2 = e_2, e_1 \cdot e_2 = e_2 \cdot e_1 = 0$ and $e_1 + e_2 = 1$ which means that e_1 and e_2 are idempotents (some times called also orthogonal idempotents). By the help of the idempotent elements e_1 and e_2 any bicomplex number

$$\xi = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 = (a_0 + i_1 a_1) + i_2(a_2 + i_1 a_3) = z_1 + i_2 z_2$$

where $a_{0,}\, a_1, a_2, a_3 \epsilon R,$

$$z_1 (= a_0 + i_1 a_1), z_2 (= a_2 + i_1 a_3) \epsilon C_1$$

can be expressed as

$$\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2$$

where $\xi_1 (= z_1 - i_1 z_2)$ and $\xi_2 (= z_1 + i_1 z_2) \epsilon C_1.$

This representation of a bicomplex number is known as the Idempotent Representation of ξ. ξ_1 and ξ_2 are called the Idempotent Components of the bicomplex number $\xi = z_1 + i_2 z_2$, resulting a pair of mutually complementary projections

$$P_1 : (z_1 + i_2 z_2) \epsilon C_2 \longmapsto (z_1 - i_1 z_2) \epsilon C_1$$

and

$$P_2 : (z_1 + i_2 z_2) \epsilon C_2 \longmapsto (z_1 + i_1 z_2) \epsilon C_1.$$

The spaces $A_1 = \{P_1(\xi) : \xi \epsilon C_2\}$ and $A_2 = \{P_2(\xi) : \xi \epsilon C_2\}$ are called the auxiliary complex spaces of bicomplex numbers.

An element $\xi = z_1 + i_2 z_2$ is singular if and only if $|z_1^2 + z_2^2| = 0$.The set of singular elements is denoted as O_2 and defined by $O_2 = \{\xi \epsilon C_2 : \xi$ is the collection of all -complex multiples of e_1 and e_2 $\}$

The norm the $|| \cdot || : C_2 \longmapsto C_0^+$(set of all non negetive real numbers) of a bicomplex number is defined as

$$||\xi|| = \sqrt{\{|z_1|^2 + |z_2|^2\}} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

2. Laplace transform

Let $f(t)$ be a real valued function of exponential order k. The coplex version of Laplace Transform [5] of $f(t)$ for $t \geq 0$ can be defined as

$$L\{f(t)\} = F_1(\xi_1) = \int_0^\infty f(t) e^{-\xi_1 t} dt$$

. Here $F_1(\xi_1 : \xi_1 \epsilon C_1)$ exists and absolutely convergent for $\text{Re}(\xi_1) > k$.Similarly

$$F_2(\xi_2) = \int_0^\infty f(t)e^{-\xi_2 t}dt$$

converges absolutely for $\text{Re}(\xi_2) > k$. Then the bicomplex Laplace Transform [4] of $f(t)$ for $t \geq 0$ can be defined as

$$L\{f(t)\} = F(\xi) = \int_0^\infty f(t)e^{-\xi t}dt$$

. Here $F(\xi)$ exists and convergent in the region

$$D = \{\xi \epsilon C_2 : \xi = \xi_1 e_1 + \xi_2 e_2 : Re(\xi_1) > k, Re(\xi_2) > k\}$$

in idempotent representation.

3. Inverse Laplace Transform for Bicomplex variables

If $f(t)$ real valued function of exponential order k, defined on $t \geq 0$,its Laplace transform $F_1(\xi_1)$ in bicomplex variable $\xi_1 = x_1 + i_1 y_1 \epsilon C_1$ is simply

$$\begin{aligned} F_1(\xi_1) &= \int_0^\infty f(t)e^{-\xi_1 t}dt = \int_0^\infty f(t)e^{-(x_1+i_1y_1)t}dt = \int_0^\infty e^{-x_1 t}f(t)e^{-i_1y_1t}dt \\ &= \int_0^\infty \{e^{-x_1 t}f(t)\}e^{-i_1y_1t}dt = \int_{-\infty}^\infty g(t)e^{-i_1y_1t}dt = \psi(x_1,y_1) \end{aligned}$$

which is Fourier transform of $g(t)$ where

$$g(t) = f(t)e^{-x_1 t}, t \geq 0; and = 0, t < 0$$

in usual complex exponential form.

$F_1(\xi_1)$ converges for $\text{Re}(\xi_1) > k$ and

$$| F_1(\xi_1)| < \infty \Rightarrow |\int_0^\infty f(t)e^{-\xi_1 t}dt| = \int_{-\infty}^\infty |g(t)e^{-i_1y_1t}|dt = \int_{-\infty}^\infty |g(t)|dt < \infty$$

The later condition shows that $g(t)$ is absotulely integrable .Then by Laplace inverse transform in complex exponential form

$$g(t) = \frac{1}{2\pi i_1}\int_{-\infty}^\infty e^{i_1y_1t}\psi(x_1,y_1)dy_1 \Rightarrow f(t) = \frac{1}{2\pi i_1}\int_{-\infty}^\infty e^{x_1t}e^{i_1y_1t}\psi(x_1,y_1)dy_1.$$

Now if we integrate along a vertical line then x_1 is a constant and so for complex variable $\xi_1 = x_1 + i_1 y_1 \epsilon C_1$(that implies $d\xi_1 = dy_1$) the above inversion formula can be

extended to complex Laplace inverse transform

$$f(t) = \frac{1}{2\pi i_1}\int_{x_1-i_1\infty}^{x_1+i_1\infty} e^{(x_1+i_1y_1)t}\psi(x_1,y_1)dy_1 = \frac{1}{2\pi i_1}\int_{x_1-i_1\infty}^{x_1+i_1\infty} e^{\xi_1 t}\psi(x_1,y_1)d\xi_1$$

$$= \frac{1}{2\pi i_1}\lim_{y_1\to\infty}\int_{x_1-i_1y_1}^{x_1+i_1y_1} e^{\xi_1 t}F(\xi_1)d\xi_1 ....(1)$$

Here the integration is to be performed along a vertical line in the complex ξ_1-plane employing contour integration method.

We assume that $F_1(\xi_1)$ is holomorphic in $x_1 < k$ except for having a finite number of poles ξ_1^k ,$k = 1, 2, 3, ...n$ therein. Taking $R \to \infty$ we can guarantee

that all these poles lie inside the contour Γ_R .Since $e^{\xi_1 t}$ never vanishes so the poles of $e^{\xi_1 t}F(\xi_1)$ and $F_1(\xi_1)$ are same.Then by Cauchy residue theorem

$$\lim_{R\to\infty}\int_{\Gamma_R} e^{\xi_1 t}F(\xi_1)d\xi_1 = 2\pi i_1 \sum \operatorname{Re}s\{e^{\xi_1 t}F(\xi_1) : \xi_1 = \xi_1^k\}.$$

Now since for ξ on C_R and $|F(\xi)\ | < \frac{M}{|\xi|^p}$ [6] some $p > 0$ and all $R > R_0$,

$$\lim_{R\to\infty}\int_{C_R} e^{\xi_1 t}F(\xi_1)d\xi_1 = 0 \ for \ t > 0$$

.

so

$$\int_{\Gamma_R} e^{\xi_1 t}F(\xi_1)d\xi_1 = \int_{C_R} e^{\xi_1 t}F(\xi_1)d\xi_1 + \int_{x_1-i_1R}^{x_1+i_1R} e^{\xi_1 t}F(\xi_1)d\xi_1 = 2\pi i_1 \sum \operatorname{Re}s\{e^{\xi_1 t}F(\xi_1) : \xi_1 = \xi_1^k\}$$

.

then for $R \to \infty$ we obtain

$$\int_{x_1-i_1\infty}^{x_1+i_1\infty} e^{\xi_1 t}F(\xi_1)d\xi_1 = 2\pi i_1 \sum \operatorname{Re}s\{e^{\xi_1 t}F(\xi_1) : \xi_1 = \xi_1^k\}, t > 0.$$

We first attend the right half plane $D_1 = \text{Re}(\xi_1) > k$ and

$$\lim_{Re(\xi_1)\longrightarrow\infty} F_1(\xi_1) = 0.$$

The inverse Laplace transform of $F_1(\xi_1)$ will then a real valued function

$$f(t)=\frac{1}{2\pi i_1}\int\limits_{x_1-i_1\infty}^{x_1+i_1\infty} e^{\xi_1 t}F_1(\xi_1)d\xi_1 \;\ldots\ldots\ldots\ldots \quad (2)$$

where $\xi_1 = x_1 + i_1 y_1 \epsilon C_1$.

In the right half plane $\mathrm{D}_2 = \mathrm{Re}(\xi_2) > k$ and

$$\lim_{Re(\xi_2)\longrightarrow\infty} F_2(\xi_2)=0$$

the inverse Laplace transform of $F_2(\xi_2)$ will be

$$f(t)=\frac{1}{2\pi i_1}\int\limits_{x_2-i_1\infty}^{x_2+i_1\infty} e^{\xi_2 t}F_2(\xi_2)d\xi_2, \xi_2 = x_2 + i_1 y_2 \epsilon C_1 \ldots\ldots\ldots\ldots(3)$$

Moreover in each case $f(t)$ is of exponential order k.

Then

$$\begin{aligned} f(t) &= f(t)e_1 + f(t)e_2 = \frac{1}{2\pi i_1}\int\limits_{D_1} e^{\xi_1 t}F_1(\xi_1)d\xi_1 e_1 + \frac{1}{2\pi i_1}\int\limits_{D_2} e^{\xi_2 t}F_2(\xi_2)d\xi_2 e_2 \\ &= \frac{1}{2\pi i_1}\int\limits_{D=D_1\cup D_2} e^{\xi t}F(\xi)d\xi \ldots\ldots\ldots(4) \end{aligned}$$

where we use the fact that any real number c can be written as

$$c = c + i_1\cdot 0 + i_2 \cdot 0 + i_1 i_2 \cdot 0 = c_1 e_1 + c_2 e_2.$$

The bicomplex version of inverse Laplace transform thus can be defined as (4). Evidently, here also

$$\lim_{Re(\xi_{1,2})\longrightarrow\infty} F(\xi)=0$$

and $f(t)$ is of exponential order k . Reversing this proces one can at once obtain f(t) from the integration defined in (4). It guarantees the existance of inverse Laplace transform.

3.1. **Definition.** If $F(\xi)$ exists and is convergent in a region $D = D_1 \cup D_2$ which are the right half planes $D_{1,2} = R(\xi_{1,2}) > k$ together with

$$\lim_{Re(\xi_{1,2}) \longrightarrow \infty} F(\xi) = 0$$

then the inverse Laplace transform of $F(\xi)$ can be defined as

$$L^{-1}\{F(\xi)\} = \frac{1}{2\pi i_1} \int\limits_{D=D_1 \cup D_2} e^{\xi t} F(\xi) d\xi = f(t)$$

The integral in each plane $D_1$and D_2 are taken along any straight line $R(\xi_{1,2}) > k$. As a result our object function $f(t)$ will be of exponential order k ,in the principal value sense.

3.2. **Examples.**

- If we take $F(\xi)d\xi = \frac{1}{\xi}$, then it's inverse Laplace transform is given by

$$f(t) = \frac{1}{2\pi i_1} \int\limits_{D=D_1 \cup D_2} e^{\xi t} F(\xi) d\xi = \frac{1}{2\pi i_1} \int\limits_{D_1} e^{\xi_1 t} F_1(\xi_1) d\xi_1 e_1 + \frac{1}{2\pi i_1} \int\limits_{D_2} e^{\xi_2 t} F_2(\xi_2) d\xi_2 e_2 \text{...........}(4)$$

Now

$$\frac{1}{2\pi i_1} \int\limits_{D_1} e^{\xi_1 t} F_1(\xi_1) d\xi_1 = \frac{1}{2\pi i_1} \int\limits_{x_1 - i_1 \infty}^{x_1 + i_1 \infty} e^{\xi_1 t} \frac{1}{\xi_1} d\xi_1 = 2\pi i_1 \cdot 1 = 2\pi i_1$$

as $\xi_1 = 0$ is the only singular point therein, so

$$residue = \lim_{\xi_1 \longrightarrow 0} (\xi - 0) e^{\xi_1 t} \frac{1}{\xi_1} = 1.$$

In a similar way,

$$\frac{1}{2\pi i_1} \int\limits_{D_2} e^{\xi_2 t} F_2(\xi_2) d\xi_2 = 2\pi i_1$$

and those leads (4) to

$$f(t) = e_1 + e_2 = 1.$$

- In our procedure one may easily check a partial list....to name a few....
- $L^{-1}\{\frac{\omega}{\xi^2 + \omega^2}\} = \sin \omega t,$
- $L^{-1}\{\frac{\xi}{\xi^2 + \omega^2}\} = \cos \omega t,$
- $L^{-1}\{\frac{\xi + a}{(\xi + a)^2 + \omega^2}\} = e^{-at} \cos \omega t,$
- $L^{-1}\{\frac{\omega}{(\xi + a)^2 + \omega^2}\} = e^{-at} \sin \omega t.$

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