Mathematical Inverse Problems, Vol. 1, No. 1 (2014), 1-7 ISSN 2381-9634

NUMERICAL SOLUTION OF FUZZY LINEAR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS BY TRIANGULAR FUNCTIONS METHOD

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Abstract. In this paper, we use parametric form of fuzzy number. Then, we convert a Volterra-Fredholm fuzzy integral equation to a system of integral equations in crisp case. A numerical method based on an m-set of general, orthogonal triangular functions (TF) is proposed to approximate the solution of linear Volterra-Fredholm fuzzy integral equations. Furthermore, a theorem is proved for convergence analysis.
2010 Mathematics Subject Classification. 45D05.
Key words and phrases. Fuzzy integral equations, Volterra-Fredholm, Triangular functions.

1. Introduction

The solutions of integral equations have a major role in the field of science and engineering. A physical even can be modelled by the differential equation [2, 3], an integral equation. Since few of these equations cannot be solved explicitly, it is often necessary to resort to numerical techniques which are appropriate combinations of numerical integration and interpolation [5, 12, 14].

The topics of fuzzy integral equations (FIE) which growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in recent years. The fuzzy mapping function was introduced by Chang and Zadeh [6]. Later, Dubois and Prade [8] presented an elementary fuzzy calculus based on the extension principle also the concept of integration of fuzzy functions was first introduced by Dubois and Prade [8]. Babolian et al., Abbasbandy et al. in [1, 4] obtained a numerical solution of linear Fredholm fuzzy integral equations of the second kind. Recently, Otadi and Mosleh presented an iterative algorithm for solving fuzzy nonlinear integral equations [15].

In this paper, we present a novel and very simple numerical method based upon orthogonal triangular functions (TF) sets for solving linear Volterra-Fredholm fuzzy integral equations.

2. Preliminaries

In this section the basic notations used in fuzzy calculus and triangular functions are introduced.

Definition 1. [10, 13] A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:

i. $\underline{u}(r)$ is a bounded monotonically increasing, left continuous function on $(0,1]$ and right continuous at 0;

ii. $\overline{u}(r)$ is a bounded monotonically decreasing, left continuous function on $(0,1]$ and right continuous at 0;

iii. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

A crisp number r is simply represented by $\underline{u}(\alpha) = \overline{u}(\alpha) = r, 0 \ \leq \alpha \leq 1$. The set of all the fuzzy numbers is denoted by E^1.

For arbitrary $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and $k \in \mathbb{R}$ we define addition and multiplication by k as

$$\begin{aligned}
&\underline{(u+v)}(r) = (\underline{u}(r) + \underline{v}(r)),\\
&\overline{(u+v)}(r) = (\overline{u}(r) + \overline{v}(r)),\\
&\underline{ku}(r) = k\underline{u}(r), \overline{ku}(r) = k\overline{u}(r),\ if\ k \geq 0,\\
&\underline{ku}(r) = k\overline{u}(r), \overline{ku}(r) = k\underline{u}(r),\ if\ k < 0.
\end{aligned}$$

Remark 1. [1] Let $u = (\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$ be a fuzzy number, we take

$$u^c(r) = \frac{\underline{u}(r) + \overline{u}(r)}{2},$$

$$u^d(r) = \frac{\overline{u}(r) - \underline{u}(r)}{2}.$$

It is clear that $u^d(r) \geq 0$, $\underline{u}(r) = u^c(r) - u^d(r)$ and $\overline{u}(r) = u^c(r) + u^d(r)$, also a fuzzy number $u \in E^1$ is said symmetric if $u^c(r)$ is independent of r for all $0 \leq r \leq 1$.

Remark 2. Let $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and also k, s are arbitrary real numbers. If $w = ku + sv$ then

$$\begin{aligned}
&w^c(r) = ku^c(r) + sv^c(r),\\
&w^d(r) = |k|u^d(r) + |s|v^d(r).
\end{aligned}$$

Definition 2. [9] For arbitrary fuzzy numbers u, v, we use the distance

$$D(u,v) = sup_{0 \leq r \leq 1} max\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\}$$

and it is shown that (E^1, D) is a complete metric space [17].

Now, we can express the m-set orthogonal triangular function (TF) vectors as

$$\mathbf{T1_m}(t) = [T1_0(t)\ T1_1(t)\ \dots\ T1_i(t) \dots\ T1_{m-1}(t)]^T$$
$$\mathbf{T2_m}(t) = [T2_0(t)\ T2_1(t)\ \dots\ T2_i(t) \dots\ T2_{m-1}(t)]^T.$$

The ith component of the vector $T1_m(t)$ is defined as

$$T1_i(t) = \begin{cases} 1 - \frac{(t-ih)}{h}, & ih \le t < (i+1)h \\ 0, & otherwise \end{cases}$$

and the ith component of the vector $T2_m(t)$ is defined as

$$T2_i(t) = \begin{cases} \frac{(t-ih)}{h}, & ih \le t < (i+1)h \\ 0, & otherwise \end{cases}$$

where $i = 0, 1, 2, \dots, (m-1)$ [7].

In general, a time function $f(t)$ of Lebesgue measure may be expanded into an m-term TF series in $t \in [0, T)$ as

$$f(t) \simeq [p_0\ \dots\ p_i\ \dots\ p_{m-1}]\mathbf{T1_m} + [q_0\ \dots\ q_i\ \dots\ q_{m-1}]\mathbf{T2_m} = P^T\mathbf{T1_m} + Q^T\mathbf{T2_m} \tag{1}$$

where, the constant coefficients are the samples of function such that $p_i = f(ih)$ and $q_i = f((i+1)h)$ where $i = 0, 1, \dots, m-1$ [7].

3. Volterra-Fredholm fuzzy integral equations

Consider the linear Volterra-Fredholm fuzzy integral equations (VFFIE) [16]

$$F(x) = G(x) + \lambda_1 \int_a^x k_1(x,t)F(t)dt + \lambda_2 \int_a^b k_2(x,t)F(t)dt \tag{2}$$

where $a \le x, t \le b$ $\lambda_1, \lambda_2 > 0$, the kernels $k_1(x,t)$ and $k_2(x,t)$ are known in $L^2(R)$ and $G(x)$ is a known fuzzy function. Without loss of generality, suppose $a = 0$ and $b = 1$. If $G(x)$ is a fuzzy function these equation may only possess fuzzy solution. Sufficient conditions for the existence of a unique solution to the VFFIE are given in [16].

Now, we introduce parametric form of a VFFIE with respect to Definition 1. Let $(\underline{G}(x;r), \overline{G}(x;r))$ and $(\underline{F}(x;r), \overline{F}(x;r))$, $0 \le r \le 1$ are parametric form of $G(x)$ and $F(x)$, respectively. Then parametric form of VFFIE is as follows:

$$\begin{array}{l} \underline{F}(x;r) = \underline{G}(x;r) + \lambda_1 \int_0^x \underline{k_1(x,t)F(t;r)}dt + \lambda_2 \int_0^1 \underline{k_2(x,t)F(t)}dt, \\ \overline{F}(x;r) = \overline{G}(x;r) + \lambda_1 \int_0^x \overline{k_1(x,t)F(t;r)}dt + \lambda_2 \int_0^1 \overline{k_2(x,t)F(t)}dt, \\ 0 \le x \le 1, \quad 0 \le r \le 1. \end{array} \tag{3}$$

By referring to Remark 2 we have

$$F^c(x;r) = G^c(x;r) + \lambda_1 \int_0^x k_1(x,t)F^c(t;r)dt + \lambda_2 \int_0^1 k_2(x,t)F^c(t;r)dt, \quad 0 \leq x \leq 1, \;\; 0 \leq r \leq 1 \tag{4}$$

$$F^d(x;r) = G^d(x;r) + \lambda_1 \int_0^x |k_1(x,t)|F^d(t;r)dt + \lambda_2 \int_0^1 |k_2(x,t)|F^d(t;r)dt, \quad 0 \leq x \leq 1, \;\; 0 \leq r \leq 1. \tag{5}$$

It is clear that we must solve two crisp Volterra-Fredholm integral equations provided that each of Eqs. (4) and (5) have solution. By using Eq. (1), we can be approximate $F^c(x;r)$ and $F^d(x;r)$ as

$$F^c(x;r) \simeq (P^c(r))^T \mathbf{T1_m}(x) + (Q^c(r))^T \mathbf{T2_m}(x) = F^c_m(x;r)$$

and

$$F^d(x;r) \simeq (P^d(r))^T \mathbf{T1_m}(x) + (Q^d(r))^T \mathbf{T2_m}(x) = F^d_m(x;r)$$

that $(P^c(r))^T = [p^c_0(r) \;\ldots\; p^c_i(r) \;\ldots\; p^c_{m-1}(r)]$ and so on. We suggest the collocation points as $s_i = ih$ where $i = 0, 1, \ldots, m-1$ and $h = \frac{T}{m}$. Now we have two system of m equations and $(m+1)$ unknowns which can be solved for the coefficients $p^c_i,\ q^c_i,\ p^d_i$ and q^d_i. Obviously, by using a conventional quadrature rules, such as Gaussian rule, we can reduce the computational efforts. A powerful search technique can be used to obtain the optimal $p^c_i,\ q^c_i,\ p^d_i$ and q^d_i with maximum validity in determining desirable approximate [7], using $MATLAB$ software.

Assume $(C[J], \|\|)$ the Banach space of all continuous functions on $J = [0,1]$ with norm $\|f(x)\| = max_{\forall x \in J}|f(x)|$. Let $|k_1(x,t)| \leq M_1$ and $|k_2(x,t)| \leq M_2$, $\forall a \leq x, t \leq b$ [11].

Theorem 1. The VFFIE (2) by using TF approximations converges if $0 < \alpha < 1$.
Proof.

$$\|F^\dagger_m - F^\dagger\| = max_{\forall x \in J} \mid F^\dagger_m(x;r) - F^\dagger(x;r) \mid \leq max_{\forall x \in J}(\mid \lambda_1 \mid \int_0^x \mid k_1(x,t) \mid\mid F^\dagger_m(x;r) - F^\dagger(x;r) \mid dt + \mid \lambda_2 \mid \int_0^1 \mid k_2(x,t) \mid \mid F^\dagger_m(x;r) - F^\dagger(x;r) \mid dt) \leq (\mid \lambda_1 \mid M_1 x + \mid \lambda_2 \mid M_2) max_{\forall x \in J} \mid F^\dagger_m(x;r) - F^\dagger(x;r) \mid = \alpha max_{\forall x \in J} \mid F^\dagger_m(x;r) - F^\dagger(x;r) \mid$$

where $\dagger$ means we have this equation for c and d together, independently. We get $(1-\alpha)\|F^\dagger_m - F^\dagger\| \leq 0$ and choose $0 < \alpha < 1$, by increasing m, it implies $\|F^\dagger_m - F^\dagger\| \longrightarrow 0$ as $m \longrightarrow \infty$. Also we have

$$max_{\forall x \in J} \mid F^c_m(x;r) - F^c(x;r) \mid \leq \alpha max_{\forall x \in J} \mid F^c_m(x;r) - F^c(x;r) \mid, \tag{6}$$

(7) $$max_{\forall x\in J} \mid F_m^d(x;r) - F^d(x;r) \mid \leq \alpha max_{\forall x\in J} \mid F_m^d(x;r) - F^d(x;r) \mid .$$

By (6), (7) and Remark 1 we have

$$\begin{aligned}
max_{\forall x\in J} \mid \underline{F_m(x;r)} - \underline{F(x;r)} \mid &\leq \alpha max_{\forall x\in J} \mid F_m^c(x;r) - F^c(x;r) \mid + \\
&\alpha max_{\forall x\in J} \mid F_m^d(x;r) - F^d(x;r) \mid, \\
max_{\forall x\in J} \mid \overline{F_m(x;r)} - \overline{F(x;r)} \mid &\leq \alpha max_{\forall x\in J} \mid F_m^c(x;r) - F^c(x;r) \mid + \\
&\alpha max_{\forall x\in J} \mid F_m^d(x;r) - F^d(x;r) \mid,
\end{aligned}$$

hence for all $r \in [0,1]$

$$\begin{gathered}
max\{max_{\forall x\in J}[\mid \underline{F_m(x;r)} - \underline{F(x;r)} \mid, \mid \overline{F_m(x;r)} - \overline{F(x;r)} \mid]\} \leq \\
\alpha max_{\forall x\in J} \mid F_m^c(x;r) - F^c(x;r) \mid + \alpha max_{\forall x\in J} \mid F_m^d(x;r) - F^d(x;r) \mid,
\end{gathered}$$

and then

$$\begin{gathered}
max_{\forall x\in J} D(F_m(x), F(x)) \leq sup_{r\in[0,1]}[\alpha max_{\forall x\in J} \mid F_m^c(x;r) - F^c(x;r) \mid + \\
\alpha max_{\forall x\in J} \mid F_m^d(x;r) - F^d(x;r) \mid]
\end{gathered}$$

and this completes the proof. $\square$

4. Numerical examples

To illustrate the technique proposed in this paper, consider the following examples.

Example 4.1. We consider the following linear Volterra-Fredholm fuzzy integral equation

$$F(x) = G(x) + \int_0^x F(t)dt + \int_0^1 (x^2 - x)F(t)dt, \quad 0 \leq x, t \leq 1,$$

where $G(x) = (r + (x^2 - x)r(1-e), (2-r) + (x^2 - x)(2-r)(1-e))$ and the exact solution in this case is given by $F(x) = (re^x, (2-r)e^x),\ 0 \leq r \leq 1$.

For this example, we consider $m = 4$. The exact and obtained solution of linear VFFIE in this example at $x = 0.5$ are shown in figure 1.

Figure 1. Compares the exact and obtained solutions with $m = 4$.

Example 4.2. We consider the following linear Volterra-Fredholm fuzzy integral equation

$$F(x) = G(x) + \int_0^x (-t)F(t)dt + \int_0^1 \frac{x}{2}F(t)dt, \quad 0 \le x, t \le 1,$$

where $G(x) = (x(r^2+r) - \frac{x}{4}(r^2+r) + \frac{x^3}{3}(4-r^3-r), x(4-r^3-r) - \frac{x}{4}(4-r^3-r) + \frac{x^3}{3}(r^2+r))$ and the exact solution in this case is given by $F(x) = ((r^2+r)x, (4-r^3-r)x),\ 0 \le r \le 1$.

For this example, we consider $m = 4$. The exact and obtained solution of linear VFFIE in this example at $x = 0.5$ are shown in figure 2.

Figure 2. Compares the exact and obtained solutions with $m = 4$.

5. Summary and conclusions

In this paper, a numerical method based on a complementary pair of orthogonal TF sets was developed to approximate the solution of linear VFFIE. The structural properties of TFs are used to reduce Volterra-Fredholm fuzzy integral equations to a system of linear equations. The approximate solutions obtained by MATLAB software are shown the validity and efficiency of the proposed method. Extensions to the case of more general nonlinear VFFIE equations are left for future studies. This paper will be utilized as a good starting point for such extensions.

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