Math. Inverse Probl. 2019 **6**:1

PARAMETER CHOICE STRATEGIES FOR WEIGHTED SIMPLIFIED REGULARIZATION METHOD FOR ILL-POSED EQUATIONS

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Received Apr. 18, 2019

ABSTRACT. In this paper, we deal with three parameter choice strategies for weighted simplified regularization method for ill-posed operator equations. Using general Holder type source condition we obtain an optimal order error estimate in all the three parameter choice strategies studied in this paper. Finally, we applied the proposed methods to an academic example to test the validity.

2010 Mathematics Subject Classification. 65J20, 65D15, 47A50.

Key words and phrases. ill-posed problem; simplified regularization; optimal order; adaptive parameter choice strategy.

1. INTRODUCTION

Numerous problems in computational sciences can be reduced to solving a Fredholm integral equation of the first kind

$$\int_a^b k(s,t)x(t)dt = y(s), \quad a \leq s \leq b,$$

where $k(.,.)$ is a non-degenerate square integrable function and $y(.)$ is a know data function. The above equation can be written as

$$Tx = y, \tag{1.1}$$

where $T : L^2[a,b] \longrightarrow L^2[a,b]$ defined by

$$(Tx)(s) = \int_a^b k(s,t)x(t)dt, \quad a \leq s \leq b.$$

Since T is compact and $k(.,.)$ is a non-degenerate [12], the range $R(T)$ of T is not closed in $L^2[a,b]$ and hence (1.1) is ill-posed [5, 12], i.e., the solution of (1.1) may not depend continuously on the data. In practice, the available data is y^δ with

$$\|y - y^\delta\| \leq \delta. \tag{1.2}$$

Therefore, one has to deal with the equation

$$Tx = y^\delta$$

instead of (1.1). Regularization methods are used for approximately solving ill-posed equations. The most famous regularization method for (1.1) is the Tikhonov regularization method, in which the minimizer x_α^δ of the functional

$$J_\alpha(x) = \|Tx - y^\delta\|^2 + \alpha\,\|x\|^2,$$

is used as an approximation for the solution $\hat{x}$ (assumed to be exist) of (1.1). Note that [1,5,12] x_α^δ satisfies the equation

$$(T^*T + \alpha I)x_\alpha^\delta = T^*\,y^\delta, \quad \alpha > 0, \tag{1.3}$$

where T^* is the adjoint of T. If the operator T in (1.1) is positive self adjoint, then the minimizer w_α^δ of the functional

$$J_\alpha(x) = \langle Tx, x\rangle - 2\,\langle y^\delta, x\rangle + \alpha\,\langle x, x\rangle \quad \alpha > 0 \tag{1.4}$$

can be used as an approximation for $\hat{x}$.

The above regularization method is known as the simplified regularization or Ritz regularization method, and it was studied by Schock in [16]. In particular, Schock [16] shown that this method has computational advantages over the Tikhonov regularization. Let w_α be the minimizer of (1.4) with y in place of y^δ. Then it is known that [3–5,15,16]

$$\|w_\alpha^\delta - w_\alpha\| \le \tfrac{\delta}{\alpha}$$

and

$$\|w_\alpha - \hat{x}\| \le \alpha^\nu, \tag{1.5}$$

provided $\hat{x} \in R(T^\nu), \quad 0 < \nu \le 1$ (Note that for $0 < \nu < 1$, we have (see [11, page 287]), $T^\nu w = \frac{\sin \pi\nu}{\pi\nu}\int_0^\infty t^\nu (T + tI)^{-2} Tw dt$).

It is known that the solution x_α^δ of (1.3) over smooths the solution $\hat{x}$ [10], and to overcome this problem two different approaches, both referred as fractional Tikhonov methods [2,8–10], have been studied. Weighted or fractional Tikhonov regularization scheme was introduced by Hochstenbach and Reichel [8]. In this method the minimizer $x_{\alpha,\beta}^\delta$ of the functional

$$J_\alpha(x) = \|Tx - y^\delta\|_\beta + \alpha\,\|x\|^2,$$

is taken as an approximation for $\hat{x}$. Here $\|x\|_\beta = \|(T\,T^*)^{(\beta-1)/4}x\|$ for some parameter $0 \le \beta \le 1$. The minimizer $x_{\alpha,\beta}^\delta$ is the solution of the normal equations [8,14]

$$((T^*T)^{(\beta+1)/2} + \alpha\,I)x = (T^*T)^{(\beta-1)/2}\,T^*\,y^\delta.$$

The aim of this paper is to study weighted or fractional simplified regularization method to approximate $\hat{x}$. Throughout this paper we assume that A is a positive semi-definite operator and we consider the operator equation

$$Ax = y, \tag{1.6}$$

where $A : H \longrightarrow H$ is a positive self adjoint operator defined on a Hilbert space H. Precisely, we study the weighted or fractional simplified regularization method, in which the minimizer $w^\delta_{\alpha,\beta}$ of the functional

$$J^\beta_\alpha(x) = \langle Ax, x\rangle - 2\langle y, x\rangle + \alpha\langle A^\beta x, x\rangle \quad \alpha > 0,$$

where $\beta \in [0, 1)$, is taken as an approximation for the solution (again denoted by) $\hat{x}$ of (1.6). The minimizer of above functional $w_{\alpha,\beta}$, satisfies the operator equation

$$(A^{1-\beta} + \alpha I)\, x = A^{-\beta} y. \tag{1.7}$$

Let $w^\delta_{\alpha,\beta}$ be the solution of

$$(A^{1-\beta} + \alpha I)\, x = A^{-\beta} Q y^\delta, \tag{1.8}$$

where Q is the orthogonal projection onto $\overline{R(A)}$.

REMARK 1.1. *We define A^{-1} as (see [6, Theorem 3.2.2]) follow. Let $\{U_\rho(x)\}$ is a net of continuous real-valued function on $[0, \|A\|]$ such that $\{xU_\rho(x)\}$ is uniformly bounded and $\lim_\rho U_\rho(x) = x^{-1}$ for $x \neq 0$ then*

$$x = \lim_\rho U_\rho(A) z$$

for all $z = Ax \in R(A)$. For example one may define $A^{-1} = \int_0^\infty e^{-Au} du$.

Note that, if $Qy^\delta \notin R(A)$, then for $Qy^\delta \in \overline{R(A)} - R(A)$, one can find $\tilde{y}^\delta \in R(A)$ such that $\|\tilde{y}^\delta - Qy^\delta\| \leq \epsilon$ for any $\epsilon > 0$. Therefore, we may take $\tilde{y}^\delta$ in place of y^δ with $\delta = \delta + \epsilon$ (because $\|\tilde{y}^\delta - y\| \leq \|\tilde{y}^\delta - Qy^\delta\| + \|Qy^\delta - y\| \leq \delta + \epsilon$), in (1.2). So without loss of generality we assume that $Qy^\delta \in R(A)$ and $A^{-\beta}Qy^\delta$ is well defined.

One of the main constrain in regularization methods is the choice of the regularization parameter α. Discrepancy principles are considered for choosing the regularization parameter.

For simplified regularization method for (1.6), in [3], the following discrepancy principle was considered

$$D(\alpha, x) := \alpha^{2p+2}\left\langle (A + \alpha I)^{-2p-2} Qx, Qx\right\rangle = c\delta^2, \;\; c > 1 \tag{1.9}$$

and in [4], the following discrepancy principle was considered

$$\|Aw^\delta_\alpha - y^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0, \quad q > 0. \tag{1.10}$$

In this study, we consider the analogues of the discrepancy principles (1.9) (see Section 3) and (1.10) (see Section 4) for weighted or fractional simplified regularization method. We also consider the adaptive parameter choice method considered by Pareversev and Schock in [13] for choosing the regularization parameter α in (1.7). Throughout this paper c, c_1, c_2, etc., denote generic constants which may take different values at different occasions.

The rest of the papers is organized as follows. In Section 2 we provide error estimates for $\|w^\delta_{\alpha,\beta} - w_{\alpha,\beta}\|$ and $\|w_{\alpha,\beta} - \hat{x}\|$. In Section 3 and Section 4 we considered the modified form of discrepancy principles (1.9) and (1.10), respectively and in Section 5 we consider the adaptive parameter choice strategy for weighted or fractional simplified regularization method. Numerical example is given in Section 6 and finally the paper ends with conclusion in Section 7.

2. Error estimates

In this Section we obtain the error estimates for $\|w^{\delta}_{\alpha,\beta} - w_{\alpha,\beta}\|$ and $\|w_{\alpha,\beta} - \hat{x}\|$ under the assumption (1.2) and

$$\hat{x} \in \{H : x = A^{\nu} z, \ \|z\| \leq \rho\}, 0 < \nu \leq 1 - \beta. \tag{2.1}$$

PROPOSITION 2.1. *Suppose y^{δ} satisfies (1.2) and $\hat{x}$ satisfies (2.1). Then*

(i) $\|w_{\alpha,\beta} - \hat{x}\| = O(\alpha^{\frac{\nu}{1-\beta}})$

and

(ii) $\|w_{\alpha,\beta} - w^{\delta}_{\alpha,\beta}\| = O(\frac{\delta}{\alpha^{\frac{1}{1-\beta}}})$.

In particular,

(iii) $\|w^{\delta}_{\alpha,\beta} - \hat{x}\| \leq c_1 \frac{\delta}{\alpha^{\frac{1}{1-\beta}}} + c_2 \, \alpha^{\frac{\nu}{1-\beta}}$.

Proof. By (1.7) and (2.1), we have

$$\begin{aligned} \|\hat{x} - w_{\alpha,\beta}\| &= \|\alpha \, (A^{1-\beta} + \alpha I)^{-1} \, \hat{x}\| \\ &= \|\alpha \, (A^{1-\beta} + \alpha I)^{-1} \, A^{\nu} \, z\| \\ &\leq \sup_{\lambda > 0} \left| \frac{\alpha \, \lambda^{v}}{(\lambda^{1-\beta} + \alpha)} \right| \, \|z\| \\ &= O(\alpha^{\frac{\nu}{1-\beta}}). \end{aligned}$$

Similarly, by (1.8) and (1.7), we have

$$\begin{aligned} \|w_{\alpha} - w^{\delta}_{\alpha,\beta}\| &= \|(A^{1-\beta} + \alpha)^{-1} \, A^{-\beta} \, Q(y - y^{\delta})\| \\ &\leq \delta \sup_{\lambda > 0} \left| \frac{\lambda^{-\beta}}{(\lambda^{1-\beta} + \alpha)} \right| \\ &= O(\frac{\delta}{\alpha^{\frac{1}{1-\beta}}}). \end{aligned}$$

Hence we proved (i) and (ii) and (iii) follows from (i) and (ii). This completes the proof.

□

REMARK 2.2. (a) *Note that, for $\beta \in [0, 1)$, $\frac{\nu}{1-\beta} > \nu$, so, we obtained better rate of convergence for $\|w_{\alpha,\beta} - \hat{x}\|$ than that of $\|w_{\alpha} - \hat{x}\|$ in (1.5).*

(b) *As already mentioned in the introduction, choosing the regularization parameter α is an important task in the regularization methods for ill-posed problems. Next three Sections are devoted for the parameter choice strategies. In Section 3 and Section 4, we considered the discrepancy principles (1.9) and (1.10) modified suitably for the weighted simplified regularization method (1.8) and in Section 5 we considered the adaptive method considered in [13] for weighted simplified regularization method (1.8).*

3. Discrepancy Principle -I

In this section we consider the discrepancy principle studied in [3] suitably modified for choosing the regularization parameter α in (1.8). For $\alpha > 0, \ \beta < p \leq 1$ and $x \in H$, let

$$D_p(\alpha, x) := \alpha^{2p+2} \left\langle (A^{1-\beta} + \alpha I)^{-2p-2} A^{-\beta} Q x, A^{-\beta} Q x \right\rangle .$$

The following lemma is used for proving our main results in this Section.

LEMMA 3.1. *For each nonzero $x \in H$, $\beta < p \leq 1$, the map $\alpha \to D_p(\alpha, x)$ is continuous, strictly increasing,*

$$\lim_{\alpha \to 0} D_p(\alpha, x) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} D_p(\alpha, x) = \|A^{-\beta} Q x\|^2.$$

In particular, if $y^\delta \notin N(A)$ and y^δ satisfies

$$\|y - y^\delta\| < \delta < \|A^{-\beta} Q y^\delta\| / \sqrt{c} \tag{3.1}$$

for some $c > 1$, then the equation

$$D_p(\alpha, y^\delta) = c\delta^2 \tag{3.2}$$

has a unique solution $\alpha = \alpha(\delta)$ such that $\alpha(\delta) \to 0$ as $\delta \to 0$.

Proof. Let $\{E_\lambda\}$ be the spectral family of the operator A. Then we have

$$D_p(\alpha, x) = \int (\alpha^{2p+2} \lambda^{-2\beta}) / (\lambda^{1-\beta} + \alpha)^{2p+2} \, d\langle E_\lambda x, x \rangle .$$

Note that the map $\alpha \to f_p(\alpha, \lambda) = (\alpha^{2p+2} \lambda^{-2\beta}) / (\lambda^{1-\beta} + \alpha)^{2p+2}$ is strictly increasing for each $\lambda > 0$, and satisfies $f_p(\alpha, \lambda) \to 0$ as $\alpha \to 0$ and $f_p(\alpha, \lambda) \to \lambda^{-2\beta}$ as $\alpha \to \infty$. Hence the result follows from the Dominated convergence theorem and by the intermediate value theorem the equation (3.2) has a unique solution $\alpha = \alpha(\delta)$. Proof of $\alpha(\delta) \to 0$ as $\delta \to 0$ follows as in [15, Lemma 1].

□

LEMMA 3.2. *Suppose that $y \neq 0$, y^δ satisfies (3.1), $c_3 = (\sqrt{c} - 1)^2$, $c_4 = (\sqrt{c} + 1)^2$ and $\alpha = \alpha(\delta)$ is chosen according to (3.2). Then*

$$c_3 \, \delta^2 \leq D_p(\alpha(\delta), y) \leq c_4 \, \delta^2.$$

Proof. For $\alpha > 0$, $\beta < p \leq 1$, let $B_\alpha = \alpha^{p+1} (A^{1-\beta} + \alpha I)^{-p-1}$. Then for each nonzero $x \in H$, we have $\|B_\alpha A^{-\beta} Q x\|^2 = D_p(\alpha, x)$. Therefore,

$$\begin{aligned} D_p(\alpha, y)^{\frac{1}{2}} &= \|B_\alpha A^{-\beta} y\| \\ &\geq \|B_\alpha A^{-\beta} Q y^\delta\| - \|B_\alpha A^{-\beta} Q (y - y^\delta)\| \\ &\geq \sqrt{c}\, \delta - \delta, \end{aligned}$$

and

$$\begin{aligned} D_p(\alpha, y)^{\frac{1}{2}} &= \|B_\alpha A^{-\beta} y\| \\ &\leq \|B_\alpha A^{-\beta} Q y^\delta\| + \|B_\alpha A^{-\beta} Q (y - y^\delta)\| \\ &\leq \sqrt{c}\, \delta + \delta. \end{aligned}$$

This completes the proof.

□

THEOREM 3.3. *Let $y \neq 0$, y^{δ} satisfies (3.1), $\hat{x}$ satisfies (2.1) and $\alpha = \alpha(\delta)$ is chosen according to (3.2). Then $w^{\delta}_{\alpha(\delta),\beta} \to \hat{x}$ as $\delta \to 0$.*

Proof. By (1.7) we have,

$$\|\hat{x} - w_{\alpha,\beta}\| = \|\alpha\,(A^{1-\beta} + \alpha\, I)^{-1}\hat{x}\| = \|R_{\alpha,\beta}\hat{x}\|, \tag{3.3}$$

where $R_{\alpha,\beta} = \alpha\,(A^{1-\beta} + \alpha\, I)^{-1}$. Then in order to prove the Theorem, by (3.3) and (ii) of Proposition 2.1, it is enough to prove that

$$(1) R_{\alpha(\delta),\beta}\,\hat{x} \to 0 \text{ as } \delta \to 0.$$

and

$$(2) \tfrac{\delta}{\alpha^{1/(1-\beta)}} \to 0 \text{ as } \delta \to 0.$$

Note that $\|R_{\alpha,\beta}\| \leq 1$ for all $\alpha > 0$ and for every $u \in R(A)$,

$$\begin{aligned}
\|R_{\alpha,\beta}\, u\| &= \|R_{\alpha,\beta}\, Av\| \\
&\leq \sup_{\lambda>0} \left| \frac{\alpha\,\lambda}{\lambda^{1-\beta} + \alpha} \right| \|v\| \\
&\leq c_4 \alpha^{\frac{1}{1-\beta}}\, \|v\|
\end{aligned}$$

for some $v \in H$. Therefore $R_{\alpha,\beta}\, u \to 0$ as $\alpha \to 0$ for every u in a dense subspace of the Hilbert space $N(A)^{\perp}$ and as a consequence of the uniform boundedness principle we obtain (1). To prove (2) let

$$C_{\alpha} = \alpha^{p}\,(A^{1-\beta} + \alpha\, I)^{-p-1}\, A^{1-\beta}, \quad \alpha > 0.$$

Then for all $u \in R(A^{p})$,

$$\begin{aligned}
\|C_{\alpha}\, u\| &= \|C_{\alpha}\, A^{p}\, v\| \\
&= \alpha^{p} \|(A^{1-\beta} + \alpha\, I)^{-p-1}\, A^{1-\beta}\, A^{p}\, v\| \\
&\leq \alpha^{p} \sup_{\lambda>0} \left| \frac{\lambda^{1-\beta+p}}{(\lambda^{1-\beta} + \alpha)^{p+1}} \right| \|v\| \\
&\leq c_4 \alpha^{\frac{p}{1-\beta}}\, \|v\|
\end{aligned}$$

for some $v \in H$. Since $\|C_{\alpha}\| \leq 1$ for all $\alpha > 0$ and $R(A^{p})$ is dense in $N(A)^{\perp}$, by the uniform boundedness principle, we obtain $C_{\alpha(\delta)}\, x \to 0$ as $\delta \to 0$. Now by Lemma 3.2,

$$\begin{aligned}
c_3\, \delta^{2} &\leq D_{p}(\alpha, y) \\
&= \alpha^{2p+2} \left\langle (A^{1-\beta} + \alpha I)^{-2p-2} A^{-\beta} y, A^{-\beta} y \right\rangle \\
&= \alpha^{2p+2} \left\langle (A^{1-\beta} + \alpha I)^{-2p-2} A^{-\beta}\, A\hat{x}, A^{-\beta}\, A\hat{x} \right\rangle \\
&= \alpha^{2p+2} \left\langle (A^{1-\beta} + \alpha I)^{-2p-2} A^{2(1-\beta)}\hat{x}, \hat{x} \right\rangle \\
&= \alpha^{2}\, \|C_{\alpha}\, \hat{x}\|^{2} \\
&\leq c_4^{2} \alpha^{\frac{2+2(p-\beta)}{1-\beta}}\, \|v\|^{2}.
\end{aligned}$$

Since $p > \beta$, we have,

$$\frac{\delta^2}{\alpha^{\frac{2}{1-\beta}}} \leq \frac{c_4^2\|v\|^2}{c_3}\alpha^{\frac{2(p-\beta)}{1-\beta}} \to 0 \quad as \quad \delta \to 0,$$

this proves (2).

□

LEMMA 3.4. *Let $y \neq 0$, y^δ satisfies (3.1), $\hat{x}$ satisfies (2.1) and $\alpha = \alpha(\delta)$ be chosen according to (3.2). Then, we have the following:*

(i)

$$\alpha = O(\delta^{\frac{1}{p+1}})$$

(ii)

$$\frac{\delta}{\alpha^{\frac{1}{1-\beta}}} = O(\delta^{\frac{\nu-\beta}{1-\beta+\nu}}), \beta \in [0, \nu).$$

Proof. By Lemmas 3.2, for all sufficiently small $\alpha > 0$, we have

$$\begin{aligned} c_2\,\delta^2 &\geq D_p(\alpha, y) \\ &= \alpha^{2p+2}\|(A^{1-\beta} + \alpha I)^{-p-1}A^{-\beta}y\|^2 \\ &\geq \alpha^{2p+2}\frac{\|A^{-\beta}y\|^2}{\|A^{1-\beta} + \alpha I\|^{2(p+1)}} \\ &\geq c_5\,\alpha^{2p+2}, \end{aligned}$$

for some constant c_5. Thus $\alpha = O(\delta^{1/(p+1)})$, this proves (i).

By (2.1), there exists $z \in H$ such that $\hat{x} = A^\nu z$, so that $y = A\hat{x} = A^{1+\nu}z$. Therefore by Lemma 3.2, we have

$$\begin{aligned} c_1\,\delta^2 &\leq D_p(\alpha, y) \\ &= \alpha^{2p+2}\left\langle (A^{1-\beta} + \alpha I)^{-2p-2}A^{-\beta}y, A^{-\beta}y \right\rangle \\ &= \alpha^{2p+2}\left\langle (A^{1-\beta} + \alpha I)^{-2p-2}A^{-\beta}\,A^{1+\nu}z, A^{-\beta}\,A^{1+\nu}z \right\rangle \\ &= \alpha^{2p+2}\|(A^{1-\beta} + \alpha I)^{-p-1}A^{1+\nu-\beta}z\|^2 \\ &\leq \alpha^{2p+2}\sup_{\lambda>0}\left|\frac{\lambda^{2(1+\nu-\beta)}}{(\lambda^{1-\beta} + \alpha)^{2(p+1)}}\right| \\ &= O(\alpha^{\frac{2(1+\nu-\beta)}{1-\beta}}). \end{aligned}$$

Hence for $\beta \in [0, \nu)$, we have $\delta = O(\alpha^{\frac{1+\nu-\beta}{1-\beta}})$, this proves (ii).

□

Combining the results in Proposition 2.1 and Lemma 3.4, we have the following Theorem.

THEOREM 3.5. *Let y^δ satisfy (3.1), $\alpha = \alpha(\delta)$ chosen according to (3.2) and $\hat{x}$ satisfies (2.1). Then, for $\beta \in [0, \nu)$*

$$\|\hat{x} - w_{\alpha,\beta}^\delta\| = O(\delta^{\frac{\nu-\beta}{(1+p)\,(1-\beta)}}).$$

□

4. Discrepancy Principle -II

In this section we consider the discrepancy principle studied in [4], suitably modified for choosing the regularization parameter α in (1.8). Precisely, for given $r > 0, q > 0$, we choose α such that

$$\|A^{-\beta}(Aw^{\delta}_{\alpha,\beta} - Qy^{\delta})\| = \frac{\delta^r}{\alpha^q}. \tag{4.1}$$

Let

$$\phi(\alpha) = \alpha^{2q}\,\|A^{-\beta}(Aw^{\delta}_{\alpha,\beta} - Qy^{\delta})\|^2, \quad \alpha > 0.$$

LEMMA 4.1. *The function $\phi(\alpha)$ is continuous and strictly increasing for $\alpha > 0$, and satisfies $\lim_{\alpha\to 0}\phi(\alpha) = 0$ and $\lim_{\alpha\to\infty}\phi(\alpha) = \infty$. In particular, there exists a unique $\alpha = \alpha(\delta)$ satisfying (4.1). Further $\alpha(\delta) \to 0$ as $\delta \to 0$.*

Proof. Observe that

$$\begin{aligned}\phi(\alpha) &= \alpha^{2q}\,\|A^{-\beta}(Aw^{\delta}_{\alpha,\beta} - Qy^{\delta})\|^2, \quad \alpha > 0\\ &= \alpha^{2q}\,\|A^{-\beta}(A\,(A^{1-\beta} + \alpha\, I)^{-1}\,A^{-\beta}Qy^{\delta} - Qy^{\delta})\|^2\\ &= \alpha^{2q}\,\|\alpha\, A^{-\beta}\,(A^{1-\beta} + \alpha\, I)^{-1}Qy^{\delta}\|^2\\ &= \alpha^{2q}\int_0^{\|A\|}\left(\frac{\alpha\,\lambda^{-\beta}}{\lambda^{1-\beta}+\alpha}\right)^2 d\langle E_\lambda Qy^{\delta}, Qy^{\delta}\rangle,\end{aligned}$$

where E_λ is spectral family of A.

Note that the map $\alpha \to f(\alpha,\lambda) = \alpha^2\,\lambda^{-2\beta}/(\lambda^{1-\beta}+\alpha)^2$ is strictly increasing. Thus $\phi(\alpha)$ is continuous, $\phi(\alpha) \to 0$ as $\alpha \to 0$, $\phi(\alpha) \to \infty$ as $\alpha \to \infty$ and $\phi(\alpha)$ is strictly increasing for $\alpha > 0$. By the intermediate value theorem the equation (4.1) has a unique solution $\alpha = \alpha(\delta)$. Now, using the arguments similar to the ones in [15, Lemma 1], one can prove $\alpha(\delta) \to 0$ as $\delta \to 0$.

□

THEOREM 4.2. *If $\alpha = \alpha(\delta)$ is chosen according to (4.1), then $\alpha = O(\delta^{\frac{r}{q+1}})$. If, in addition, $r \le (q+1)(1-\beta)$, then $\frac{\delta}{\alpha^{1/(1-\beta)}} = O(\delta^m)$, $m = 1 - \frac{r}{(q+1)(1-\beta)}$, and $w^{\delta}_{\alpha,\beta} \to \hat{x}$ as $\delta \to 0$.*

Proof. Note that

$$\begin{aligned}\|A^{-\beta}Qy^{\delta}\| - \frac{\delta^r}{\alpha^q} &= \|A^{-\beta}Qy^{\delta}\| - \|A^{1-\beta}\,w^{\delta}_{\alpha,\beta} - A^{\beta}Qy^{\delta}\|\\ &\le \|A^{1-\beta}\,w^{\delta}_{\alpha,\beta}\|\\ &= \frac{\|A^{1-\beta}(A^{1-\beta}\,w^{\delta}_{\alpha,\beta} - A^{-\beta}Qy^{\delta})\|}{\alpha}\\ &\le \|A^{1-\beta}\|\,\frac{\delta^r}{\alpha^{q+1}},\end{aligned}$$

so,

$$\begin{aligned}\|A^{-\beta}Qy^{\delta}\| &\le \frac{\delta^r}{\alpha^q}\,(1 + \frac{\|A^{1-\beta}\|}{\alpha})\\ &\le \frac{\delta^r}{\alpha^{q+1}}\,(\alpha + \|A^{1-\beta}\|)\\ \alpha^{q+1} &\le \delta^r\,\frac{(\alpha + \|A^{1-\beta}\|)}{\|A^{-\beta}Qy^{\delta}\|}.\end{aligned}$$

This implies $\alpha = O(\delta^{r/(q+1)})$.

Further, note that

$$\frac{\delta^r}{\alpha^q} = \|A^{1-\beta} w^\delta_{\alpha,\beta} - A^\beta Q y^\delta\| = \|\alpha\, w^\delta_{\alpha,\beta}\| \le \alpha\,(\|w^\delta_{\alpha,\beta} - w_{\alpha,\beta}\| + \|w_{\alpha,\beta}\|). \tag{4.2}$$

But by Proposition 2.1,

$$\|w_{\alpha,\beta} - w^\delta_{\alpha,\beta}\| = O(\frac{\delta}{\alpha^{\frac{1}{\beta+1}}})$$

and $\|w_{\alpha,\beta}\| = \|(A^{1-\beta} + \alpha\, I)^{-1}\, A^{1-\beta}\,\hat{x}\| \le \|\hat{x}\|$. Therefore, we have

$$\frac{\delta^r}{\alpha^q} \le \alpha\,(c_2 \frac{\delta}{\alpha^{1/\beta+1}} + \|\hat{x}\|) = c_2 \alpha^{\frac{\beta}{\beta+1}}\,\delta + \alpha\,\|\hat{x}\|.$$

Now using the estimate $\alpha = O(\delta^{\frac{r}{q+1}})$, we get

$$\begin{aligned}
\frac{\delta}{\alpha^{\frac{1}{1-\beta}}} &= \delta^{1-\frac{r}{q(1-\beta)}}\,(\frac{\delta^r}{\alpha^q})^{1/q(1-\beta)} \\
&\le \delta^{1-\frac{r}{q(1-\beta)}}\,(c_2 \alpha^{\frac{\beta}{1-\beta}}\,\delta + \alpha\,\|\hat{x}\|)^{1/q(1-\beta)} \\
&\le (c_2 \delta^{1+(1-\beta)q-r}\,\alpha^{\frac{\beta}{1-\beta}} + c_6\,\delta^{(1-\beta)q-r+\frac{r}{q+1}})^{1/q(1-\beta)} \\
&\le (c_7\,\delta^{1+(1-\beta)q-r+\frac{r\beta}{(q+1)(1-\beta)}} + c_6\,\delta^{(1-\beta)q-r+\frac{r}{q+1}})^{1/q(1-\beta)} \\
&= O(\delta^{1-\frac{r}{(q+1)(1-\beta)}}) \\
&= O(\delta^m)
\end{aligned}$$

where $m = 1 - \frac{r}{(q+1)(1-\beta)}$. So $w^\delta_{\alpha,\beta} \longrightarrow \hat{x}$ follows as in Theorem 3.3.

□

THEOREM 4.3. *Let $\hat{x}$ satisfies (2.1), $q > 0, r \le (q+1)(1-\beta)$ and $\alpha = \alpha(\delta)$ be chosen according to (4.1). Then*

$$\textit{(i)} \qquad \|\hat{x} - w^\delta_{\alpha,\beta}\| = O(\delta^s),$$

where $s = min\left\{\frac{r\nu}{(q+1)(1-\beta)}, 1 - \frac{r}{(q+1)(1-\beta)}\right\}$. For a fixed ν the best rate is obtained when $r = \frac{(q+1)(1-\beta)}{\nu+1}$ which gives $\alpha = O(\delta^{\frac{1-\beta}{\nu+1}})$ and

$$\textit{(ii)} \qquad \|\hat{x} - w^\delta_{\alpha,\beta}\| = O(\delta^{\frac{\nu}{\nu+1}}).$$

Proof. From Proposition 2.1, we have

$$\|\hat{x} - w^\delta_{\alpha,\beta}\| \le c_2 \alpha^{\frac{\nu}{1-\beta}} + c_1 \frac{\delta}{\alpha^{\frac{1}{\beta+1}}},$$

so that the result in (i) follows from Theorem 4.2. If $r = \frac{(q+1)(1-\beta)}{\nu+1}$ then $\frac{r\nu}{(q+1)(1-\beta)} = 1 - \frac{r}{(q+1)(\beta+1)}$ so that $O(\alpha^{\frac{\nu}{1-\beta}}) = O(\frac{\delta}{\alpha^{\frac{1}{1-\beta}}}) = O(\delta^{\frac{\nu}{\nu+1}})$, proving (ii).

□

REMARK 4.4. (1) *Note that we obtained the optimal rate $O(\delta^{\frac{v}{v+1}})$, by choosing $\frac{r}{q+1} = \frac{(1-\beta)}{v+1}$.*

(2) *The discrepancy principle-I and discrepancy principle-II considered in Section 3 and in Section 4, can achieve the so-called better rates only when p, q and r are chosen depending on ν in the source condition. Unfortunately this ν is difficult to know in practical applications. So, we consider the adaptive selection of parameter, which is independent of ν, considered by Pereverzev and Schock in [13] in the next section.*

5. Adaptive selection of the parameter

Note that by (iii) of Proposition 2.1, we have

$$\|w^{\delta}_{\alpha,\beta} - \hat{x}\| \leq C(\frac{\delta}{\alpha^{\frac{1}{1-\beta}}} + \alpha^{\frac{\nu}{1-\beta}}) \tag{5.1}$$

where

$$C = \max\{c_1, c_2\}. \tag{5.2}$$

Further observe that the error $\frac{\delta}{\alpha^{\frac{1}{1-\beta}}} + \alpha^{\frac{\nu}{1-\beta}}$ in (5.1) is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies, $\frac{\delta}{\alpha^{\frac{1}{1-\beta}}} = \alpha^{\frac{\nu}{1-\beta}}$. That is $\alpha_\delta = \delta^{\frac{1-\beta}{\nu+1}}$. In order to obtain the optimal order in (5.1), Pereverzev and Schock in [13], introduced the adaptive selection of the parameter strategy, we modified adaptive method suitably for the situation for choosing the parameter α. Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^i \alpha_0$ where $\mu > 1$ and $\alpha_0 > \delta$.

Let

$$l := max\left\{i : \alpha_i^{\frac{\nu}{1-\beta}} \leq \frac{\delta}{\alpha_i^{\frac{1}{1-\beta}}}\right\} < N \text{ and} \tag{5.3}$$

$$k := max\left\{i : \|w^{\delta}_{\alpha_i,\beta} - w^{\delta}_{\alpha_j,\beta}\| \leq 4C\frac{\delta}{\alpha_j^{\frac{1}{1-\beta}}}, j = 0, 1, 2, \cdots, i-1\right\} \tag{5.4}$$

where $C = \max\{c_1, c_2\}$ where c_1, c_2 is as in Proposition 2.1. Now we have the following Theorem.

THEOREM 5.1. *Assume that there exists $i \in \{0, 1, \cdots, N\}$ such that $\alpha_i^{\frac{\nu}{1-\beta}} \leq \frac{\delta}{\alpha_i^{\frac{1}{1-\beta}}}$. Let assumptions of Proposition 2.1 be fulfilled, and let l and k be as in (5.3) and (5.4) respectively. Then $l \leq k$; and*

$$\|w^{\delta}_{\alpha_k,\beta} - \hat{x}\| \leq 6C\mu^{\frac{\nu+1}{1-\beta}}\delta^{\frac{\nu}{\nu+1}}.$$

Proof. To prove $l \leq k$, it is enough to show that, for each $i \in \{1, 2, \ldots N\}$, $\alpha_i^{\frac{\nu}{1-\beta}} \leq \frac{\delta}{\alpha_i^{\frac{1}{1-\beta}}} \implies \|w^{\delta}_{\alpha_i,\beta} - w^{\delta}_{\alpha_j,\beta}\| \leq 4C\frac{\delta}{\alpha_j^{\frac{1}{1-\beta}}}$, $\forall j = 0, 1, 2, \ldots i-1$. For $j < i$, we have

$$\begin{aligned}
\| w^{\delta}_{\alpha_i,\beta} - w^{\delta}_{\alpha_j,\beta} \| &\leq \| w^{\delta}_{\alpha_i,\beta} - \hat{x} \| + \| \hat{x} - w^{\delta}_{\alpha_j,\beta} \| \\
&\leq C(\alpha_i^{\frac{\nu}{1-\beta}} + \frac{\delta}{\alpha_i^{\frac{1}{1-\beta}}}) + C(\alpha_j^{\frac{\nu}{1-\beta}} + \frac{\delta}{\alpha_j^{\frac{1}{1-\beta}}}) \\
&\leq 2C\alpha_i^{\frac{\nu}{1-\beta}} + 2C\frac{\delta}{\alpha_j^{\frac{1}{1-\beta}}} \\
&\leq 4C\frac{\delta}{\alpha_j^{\frac{1}{1-\beta}}}.
\end{aligned}$$

Thus the relation $l \leq k$ is proved. Further note that

$$\| \hat{x} - w^{\delta}_{\alpha_k,\beta} \| \leq \| \hat{x} - w^{\delta}_{\alpha_l,\beta} \| + \| w^{\delta}_{\alpha_l,\beta} - w^{\delta}_{\alpha_k,\beta} \|$$

where

$$\| \hat{x} - w^{\delta}_{\alpha_l,\beta} \| \leq C(\alpha_l^{\frac{\nu}{1-\beta}} + \frac{\delta}{\alpha_l^{\frac{1}{1-\beta}}}) \leq 2C\frac{\delta}{\alpha_l^{\frac{1}{1-\beta}}}.$$

Now since $l \le k$, we have

$$\| w^{\delta}_{\alpha_k,\beta} - w^{\delta}_{\alpha_l,\beta} \| \quad \le \quad 4C\frac{\delta}{\alpha_l^{\frac{1}{1-\beta}}}.$$

Hence

$$\| \hat{x} - w^{\delta}_{\alpha_k,\beta} \| \le 6C\frac{\delta}{\alpha_l^{\frac{1}{1-\beta}}}$$

Again, since $\alpha_\delta^{\frac{\nu+1}{1-\beta}} = \delta \le \alpha_{l+1}^{\frac{\nu+1}{1-\beta}} \le \mu^{\frac{\nu+1}{1-\beta}} \alpha_l{}^{\frac{\nu+1}{1-\beta}}$, it follows that

$$\frac{\delta}{\alpha_\delta^{\frac{1}{1-\beta}}} \le \frac{\delta}{\alpha_l^{\frac{1}{1-\beta}}} \le \mu^{\frac{\nu+1}{1-\beta}} \alpha_l{}^{\frac{\nu}{(1-\beta)}} \le \mu^{\frac{\nu+1}{1-\beta}} \alpha_\delta{}^{\frac{\nu}{(1-\beta)}} \le \mu^{\frac{\nu+1}{1-\beta}} \delta^{\frac{\nu}{\nu+1}}.$$

This completes the proof.

5.1. **Implementation of adaptive choice rule.** Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 5.1 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \cdots, N$.

5.2. **Algorithm.**

1. Set $i = 0$.
2. Solve $w^{\delta}_{\alpha_i,\beta}$ by using (3.2).
3. If $\|w^{\delta}_{\alpha_i,\beta} - w^{\delta}_{\alpha_j,\beta}\| > 4C\frac{\delta}{\alpha_j^{\frac{1}{1-\beta}}}, j = 0, 1, 2, \dots i-1$, then take $k = i-1$ and return $w_{\alpha_k,\beta}$.
4. Else set $i = i + 1$ and go to 2.

6. Numerical examples

In this section, we consider an academic example for the numerical discussion to validate our theoretical results. The discrete version of the operator A is taken from the Regularization Toolbox by Hansen [7].

We adopted the Newton's method to solve above nonlinear equations (3.2) and (4.1) for α with different values β, δ, p, r and q with $q = r - 1$. Relative errors $E_{\alpha,\beta} := \left(\frac{\|x^{\delta}_{\alpha,\beta} - x^{\dagger}\|}{\|x^{\dagger}\|}\right)$, and α are presented in the tables for different values of β, p, r, n (size of the mesh) and noise level δ.

Example 4.2 Let

$$[Tx](s) := \int_{-\pi}^{\pi} k(s,t)x(t)dt = g(s), \ -\pi \le s \le \pi, \tag{6.1}$$

where $k(s,t) = (cos(s) + cos(t))^2(\frac{sin(u)}{u})^2$, $u = \pi(sin(s) + sin(t))$. We take $A := T^*T$ and $y = T^*g$ for our computation. The solution $x^{\dagger}$ is given by $x(t)^{\dagger} = a_1 exp(-c_1(t-t_1))^2) + a_2 exp(-c_2(t-t_2))^2)$. We have introduced the random noise level $\delta = 0.05$ and 0.01 in the exact data. Relative errors and α values are showcased in Tables 1–3 obtained using discrepancy principle-I, discrepancy principle-II, and the adaptive method respectively, for different values of β, p, r, n and δ.

TABLE 1. Relative errors for discrepancy principle-I.

β		$n=100$		$n=500$		$n=1000$	
		$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$
0	α	$1.038235e-01$	$5.633752e-02$	$5.709557e-02$	$2.983508e-02$	$4.201687e-02$	$2.288945e-02$
	$E_{\alpha,\beta}$	$1.846841e-01$	$1.777739e-01$	$1.802147e-01$	$1.670909e-01$	$1.683492e-01$	$1.637891e-01$
	p	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$
0	α	$9.034614e-02$	$4.018162e-02$	$4.900657e-02$	$2.153913e-02$	$3.784975e-02$	$1.482163e-02$
	$E_{\alpha,\beta}$	$1.867070e-01$	$1.707974e-01$	$1.737618e-01$	$1.622070e-01$	$1.696070e-01$	$1.573365e-01$
	p	$2/3$	$2/3$	$2/3$	$2/3$	$2/3$	$2/3$
0	α	$6.318497e-02$	$9.587215e-03$	$3.307382e-02$	$4.056634e-03$	$2.709670e-02$	$2.331297e-03$
	$E_{\alpha,\beta}$	$1.849394e-01$	$1.481517e-01$	$1.675791e-01$	$1.365604e-01$	$1.670293e-01$	$1.182621e-01$
	p	1	1	1	1	1	1
0.15	α	$1.121729e-01$	$5.931568e-02$	$5.762544e-02$	$3.140849e-02$	$4.407745e-02$	$2.381447e-02$
	$E_{\alpha,\beta}$	$1.928669e-01$	$1.718287e-01$	$1.643411e-01$	$1.573068e-01$	$1.595378e-01$	$1.523359e-01$
	p	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$
0.15	α	$9.368723e-02$	$4.350537e-02$	$5.386323e-02$	$2.154639e-02$	$3.938599e-02$	$1.351661e-02$
	$E_{\alpha,\beta}$	$1.807037e-01$	$1.641363e-01$	$1.695588e-01$	$1.483044e-01$	$1.603532e-01$	$1.370803e-01$
	p	$2/3$	$2/3$	$2/3$	$2/3$	$2/3$	$2/3$
0.15	α	$6.392676e-02$	$7.187664e-03$	$3.210264e-02$	$4.068169e-03$	$1.954892e-02$	$2.769513e-03$
	$E_{\alpha,\beta}$	$1.705366e-01$	$1.104975e-01$	$1.571158e-01$	$1.028908e-01$	$1.428269e-01$	$8.778747e-02$
	p	1	1	1	1	1	1
0.25	α	$1.027793e-01$	$5.891488e-02$	$5.830293e-02$	$3.198677e-02$	$4.494245e-02$	$2.389980e-02$
	$E_{\alpha,\beta}$	$1.590265e-01$	$1.609385e-01$	$1.590501e-01$	$1.475253e-01$	$1.531484e-01$	$1.373195e-01$
	p	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$
0.25	α	$8.613665e-02$	$4.458716e-02$	$5.541284e-02$	$2.258585e-02$	$3.180878e-02$	$1.609262e-02$
	$E_{\alpha,\beta}$	$1.508893e-01$	$1.538537e-01$	$1.638523e-01$	$1.377718e-01$	$1.208649e-01$	$1.284374e-01$
	p	$2/3$	$2/3$	$2/3$	$2/3$	$2/3$	$2/3$
0.25	α	$6.610678e-02$	$1.128888e-02$	$3.754577e-02$	$5.500000e-03$	$3.255848e-02$	$3.733918e-03$
	$E_{\alpha,\beta}$	$1.719273e-01$	$1.155649e-01$	$1.508187e-01$	$1.028954e-01$	$1.563724e-01$	$7.240657e-02$
	p	1	1	1	1	1	1
0.35	α	$1.044218e-01$	$5.891491e-02$	$5.656391e-02$	$3.188663e-02$	$4.605740e-02$	$2.391486e-02$
	$E_{\alpha,\beta}$	$1.828589e-01$	$1.537426e-01$	$1.445362e-01$	$1.266817e-01$	$1.520494e-01$	$1.175264e-01$
	p	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$
0.35	α	$8.151534e-02$	$4.521998e-02$	$5.598205e-02$	$2.376004e-02$	$3.951507e-02$	$1.718729e-02$
	$E_{\alpha,\beta}$	$1.338753e-01$	$1.401163e-01$	$1.544583e-01$	$1.219481e-01$	$1.385935e-01$	$1.110965e-01$
	p	$2/3$	$2/3$	$2/3$	$2/3$	$2/3$	$2/3$
0.35	α	$5.576333e-02$	$1.137152e-02$	$2.465876e-02$	$4.966749e-03$	$1.593763e-02$	$3.733918e-03$
	$E_{\alpha,\beta}$	$1.315137e-01$	$9.130048e-02$	$1.301862e-01$	$5.466374e-02$	$1.022811e-01$	$7.240657e-02$
	p	1	1	1	1	1	1

Table 2. Relative errors for discrepancy principle-II.

β		$n=100$		$n=500$		$n=1000$	
		$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$
0	α	$2.409365e-02$	$4.960682e-03$	$1.823065e-02$	$4.054100e-03$	$1.626371e-02$	$3.482412e-03$
	$E_{\alpha,\beta}$	$1.617371e-01$	$1.319166e-01$	$1.621281e-01$	$1.355377e-01$	$1.590535e-01$	$1.284244e-01$
	r	3	3	3	3	3	3
0.15	α	$2.388507e-02$	$4.958752e-03$	$1.821108e-02$	$4.052624e-03$	$1.625577e-02$	$3.480592e-03$
	$E_{\alpha,\beta}$	$1.260072e-01$	$1.088610e-01$	$1.378881e-01$	$1.022840e-01$	$1.457864e-01$	$9.255840e-02$
	r	3	3	3	3	3	3
0.25	α	$2.382287e-02$	$4.958898e-03$	$1.823654e-02$	$4.050538e-03$	$1.622840e-02$	$3.479817e-03$
	$E_{\alpha,\beta}$	$1.427260e-01$	$9.135510e-02$	$1.487823e-01$	$7.232072e-02$	$1.083420e-01$	$8.011626e-02$
	r	3	3	3	3	3	3
0.35	α	$1.648290e-02$	$3.568452e-03$	$1.103797e-02$	$2.563328e-03$	$9.250479e-03$	$2.022885e-03$
	$E_{\alpha,\beta}$	$1.552694e-01$	$1.355981e-01$	$8.589792e-02$	$4.765525e-02$	$8.444161e-02$	$7.593863e-02$
	r	2	2	2	2	2	2

Table 3. Relative errors obtained from Adaptive method

β		$n=100$		$n=500$		$n=1000$	
		$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$
0	α	$1.169230e-01$	$7.260000e-02$	$8.784600e-02$	$6.600000e-02$	$7.986000e-02$	$6.600000e-02$
	$E_{\alpha,\beta}$	$2.020121e-01$	$1.850613e-01$	$1.929860e-01$	$1.821091e-01$	$1.876446e-01$	$1.820621e-01$
0.15	α	$1.286153e-01$	$7.260000e-02$	$8.784600e-02$	$6.600000e-02$	$7.986000e-02$	$6.600000e-02$
	$E_{\alpha,\beta}$	$1.976656e-01$	$1.775612e-01$	$1.812653e-01$	$1.731357e-01$	$1.812824e-01$	$1.734605e-01$
0.25	α	$1.556245e-01$	$7.986000e-02$	$9.663060e-02$	$7.260000e-02$	$8.784600e-02$	$6.600000e-02$
	$E_{\alpha,\beta}$	$1.945216e-01$	$1.709314e-01$	$1.754912e-01$	$1.678725e-01$	$1.722799e-01$	$1.655920e-01$
0.35	α	$1.883057e-01$	$8.784600e-02$	$1.169230e-01$	$7.260000e-02$	$9.663060e-02$	$7.260000e-02$
	$E_{\alpha,\beta}$	$1.871009e-01$	$1.682024e-01$	$1.732220e-01$	$1.548234e-01$	$1.641372e-01$	$1.587685e-01$

7. Conclusion

In this paper we considered three parameter choice strategies for weighted simplified regularization method for ill-posed equations involving positive self adjoint operator. We obtained an optimal order error estimate under a general Holder type source condition. Numerical experiments confirms the theoretical results.

Acknowledgement

The work of Santhosh George is supported by the Core Research Grant by SERB, Department of Science and Technology, Govt. of India, EMR/2017/001594. Kanagaraj would like to thank National Institute of Technology Karnataka, India, for the financial support.

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