

INVERSE STURM-LIOUVILLE PROBLEMS WITH SUMMABLE POTENTIAL

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ABSTRACT. We describe the necessary and sufficient conditions for two sequences $\{\mu_n\}_{n=0}^\infty$ and $\{a_n\}_{n=0}^\infty$ to be correspondingly the set of eigenvalues and the set of norming constants of a Sturm-Liouville problem with real summable potential q and in advance fixed separated boundary conditions.

Keywords: Inverse Sturm-Liouville problem; eigenvalues; norming constants.

1. INTRODUCTION AND STATEMENTS OF THE RESULTS

Let us denote by $L(q, \alpha, \beta)$ the Sturm-Liouville boundary-value problem

$$(1.1) \quad \ell y \equiv -y'' + q(x)y = \mu y, \quad x \in (0, \pi), \quad \mu \in \mathbb{C},$$

$$(1.2) \quad y(0) \cos \alpha + y'(0) \sin \alpha = 0, \quad \alpha \in (0, \pi],$$

$$(1.3) \quad y(\pi) \cos \beta + y'(\pi) \sin \beta = 0, \quad \beta \in [0, \pi),$$

where q is a real-valued, summable function on $[0, \pi]$ (we write $q \in L^1_{\mathbb{R}}[0, \pi]$). By $L(q, \alpha, \beta)$ we also denote the self-adjoint operator, generated by problem (1.1)-(1.3) (see [1]). It is well-known, that under these conditions the spectra of the operator $L(q, \alpha, \beta)$ is discrete and consists of real, simple eigenvalues (see, e.g. [1, 2, 3]), which we denote by $\mu_n = \mu_n(q, \alpha, \beta) = \lambda_n^2(q, \alpha, \beta)$, $n = 0, 1, 2, \dots$, emphasizing the dependence of μ_n on q , α and β . We assume that eigenvalues are enumerated in the increasing order, i.e.,

$$\mu_0(q, \alpha, \beta) < \mu_1(q, \alpha, \beta) < \dots < \mu_n(q, \alpha, \beta) < \dots$$

In this article we consider the case $\alpha, \beta \in (0, \pi)$. It is connected with the circumstance, that in this case the principle term of asymptotics of $\lambda_n = \sqrt{\mu_n}$ is n and the principle term of asymptotics of norming constants a_n (see below (1.4) and (1.7a),(1.7b)) is $\frac{\pi}{2}$. The other three cases: 1) $\alpha = \pi, \beta \in (0, \pi)$, 2) $\alpha \in (0, \pi), \beta = 0$, 3) $\alpha = \pi, \beta = 0$, need a separate investigation and we do not concern it here.

Let $\varphi(x, \mu) = \varphi(x, \mu, \alpha, q)$ and $\psi(x, \mu) = \psi(x, \mu, \beta, q)$ are the solutions of the equation (1.1), which satisfy the initial conditions

$$\begin{aligned}\varphi(0, \mu, \alpha, q) &= 1, & \varphi'(0, \mu, \alpha, q) &= -\cot \alpha, \\ \psi(\pi, \mu, \beta, q) &= 1, & \psi'(\pi, \mu, \beta, q) &= -\cot \beta,\end{aligned}$$

respectively. The eigenvalues $\mu_n = \mu_n(q, \alpha, \beta)$, $n = 0, 1, 2, \dots$, of $L(q, \alpha, \beta)$ are the zeroes of the characteristic function

$$\Delta(\mu) := \varphi(\pi, \mu, \alpha, q) \cot \beta + \varphi'(\pi, \mu, \alpha, q) = -(\psi(0, \mu, \beta, q) \cot \alpha + \psi'(0, \mu, \beta, q)).$$

It is easy to see that functions $\varphi_n(x) := \varphi(x, \mu_n, \alpha, q)$ and $\psi_n(x) := \psi(x, \mu_n, \beta, q)$, $n = 0, 1, 2, \dots$, are the eigenfunctions, corresponding to the eigenvalue μ_n . The squares of the L^2 -norm of these eigenfunctions:

$$(1.4) \quad a_n = a_n(q, \alpha, \beta) := \int_0^\pi |\varphi_n(x)|^2 dx, \quad n = 0, 1, 2, \dots,$$

$$(1.5) \quad b_n = b_n(q, \alpha, \beta) := \int_0^\pi |\psi_n(x)|^2 dx, \quad n = 0, 1, 2, \dots,$$

are called norming constants. The sequences $\{\mu_n\}_{n=0}^\infty$, $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ are called spectral data. The famous theorem of Marchenko (see [4, 5]) asserts that two sequences $\{\mu_n\}_{n=0}^\infty$ and $\{a_n\}_{n=0}^\infty$ (or $\{\mu_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$) uniquely determine the problem $L(q, \alpha, \beta)$ ¹.

In this article we state the question:

What kind the sequences $\{\mu_n\}_{n=0}^\infty$ and $\{a_n\}_{n=0}^\infty$ should be, to be the spectral data for a problem $L(q, \alpha, \beta)$ with a $q \in L^1_{\mathbb{R}}[0, \pi]$ and in advance fixed α and β from $(0, \pi)$?

Such a question (but without the condition of fixed α and β and for different class of potential q instead of our $q \in L^1_{\mathbb{R}}[0, \pi]$) was considered first by Gelfand and Levitan in work [7] and after in many papers (we refer only some of them: [8, 9, 10]) and this problem called the inverse Sturm-Liouville problem by "spectral function" (see also, e.g. [11, 12]).

Our answer to above question is in the following assertion.

Theorem 1.1. *For a real increasing sequence $\{\lambda_n^2\}_{n=0}^\infty$ and a positive sequence $\{a_n\}_{n=0}^\infty$ to be spectral data for boundary-value problem $L(q, \alpha, \beta)$ with a $q \in L^1_{\mathbb{R}}[0, \pi]$ and fixed $\alpha, \beta \in (0, \pi)$ it is necessary and sufficient that the following relations hold:*

1) the sequence $\{\lambda_n\}_{n=0}^\infty$ has asymptotic form

$$(1.6a) \quad \lambda_n = n + \frac{\omega}{n} + l_n,$$

¹Recently Ashrafyan has found a new kind of extension of Marchenko theorem, see [6].

where $\omega = \text{const}$,

$$(1.6b) \quad l_n = o\left(\frac{1}{n}\right), \quad \text{when } n \rightarrow \infty,$$

and the function $l(\cdot)$, defined by formula

$$(1.6c) \quad l(x) = \sum_{n=1}^{\infty} l_n \sin nx,$$

is absolutely continuous on arbitrary segment $[a, b] \subset (0, 2\pi)$, i.e.

$$(1.6d) \quad l \in AC(0, 2\pi);$$

2) the sequence $\{a_n\}_{n=0}^{\infty}$ has asymptotic form

$$(1.7a) \quad a_n = \frac{\pi}{2} + s_n,$$

where

$$(1.7b) \quad s_n = o\left(\frac{1}{n}\right), \quad \text{when } n \rightarrow \infty,$$

and the function $s(\cdot)$, defined by formula

$$(1.7c) \quad s(x) = \sum_{n=1}^{\infty} s_n \cos nx,$$

is absolutely continuous on arbitrary segment $[a, b] \subset (0, 2\pi)$, i.e.

$$(1.7d) \quad s \in AC(0, 2\pi);$$

3)

$$(1.8) \quad \frac{1}{a_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{a_n} - \frac{2}{\pi} \right) = \cot \alpha,$$

4)

$$(1.9) \quad \frac{a_0}{\left(\pi \prod_{k=1}^{\infty} \frac{\mu_k - \mu_0}{k^2}\right)^2} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{a_n n^4}{\left(\pi [\mu_0 - \mu_n] \prod_{k=1, k \neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2}\right)^2} - \frac{2}{\pi} \right) = -\cot \beta.$$

In what follows, under condition (1.6) we understand the conditions (1.6a)–(1.6d) and under condition (1.7) the conditions (1.7a)–(1.7d).

To prove Theorem 1.1 we use the following assertion, which has independent interest.

Theorem 1.2. *Let $q \in L^1_{\mathbb{R}}[0, \pi]$ and $\alpha, \beta \in (0, \pi)$. Then for norming constants $a_n = a_n(q, \alpha, \beta)$ and $b_n = b_n(q, \alpha, \beta)$ the following relations are valid*

$$(1.10) \quad \frac{1}{a_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{a_n} - \frac{2}{\pi} \right) = \cot \alpha,$$

$$(1.11) \quad \frac{1}{b_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{b_n} - \frac{2}{\pi} \right) = -\cot \beta.$$

Similar results we have obtained in [13] for the case $q \in L^2_{\mathbb{R}}[0, \pi]$. In that case instead of (1.6) we had

$$(1.12) \quad \lambda_n = n + \frac{\omega}{n} + l_n, \quad \text{where} \quad l_n = \frac{\omega_n}{n}, \quad \{\omega_n\}_{n=0}^{\infty} \in l^2,$$

and instead of (1.7) we had

$$(1.13) \quad a_n = \frac{\pi}{2} + s_n, \quad \text{where} \quad s_n = \frac{\kappa_n}{n}, \quad \{\kappa_n\}_{n=0}^{\infty} \in l^2.$$

The aim of this paper is to show that when we change the condition $q \in L^2_{\mathbb{R}}[0, \pi]$ to $q \in L^1_{\mathbb{R}}[0, \pi]$, we must change (1.12) by (1.6) and (1.13) by (1.7). We should say, that the asymptotics (1.6) and (1.7) have the roots in paper of Zhikov [9]. Also we must note that conditions (1.9) and (1.11) are equivalent. It is a corollary of the fact, that norming constants $b_n = b_n(q, \alpha, \beta)$, $n = 0, 1, 2, \dots$, (see (1.5)) can be represented by spectral data $\{\mu_n\}_{n=0}^{\infty}$ and norming constants $\{a_n\}_{n=0}^{\infty}$ by the formulae (see [13])

$$(1.14) \quad \frac{1}{b_0} = \frac{a_0}{\pi^2 \left(\prod_{k=1}^{\infty} \frac{\mu_k - \mu_0}{k^2} \right)^2},$$

$$(1.15) \quad \frac{1}{b_n} = \frac{a_n n^4}{\pi^2 [\mu_0 - \mu_n]^2 \left(\prod_{k=1, k \neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2} \right)^2}, \quad n = 1, 2, \dots$$

In the same time Theorem 1.2 stay valid if we change the case $q \in L^2_{\mathbb{R}}[0, \pi]$ to the case $q \in L^1_{\mathbb{R}}[0, \pi]$

The inverse Sturm-Liouville problem with fixed α and β from $(0, \pi)$ and $q \in L^2_{\mathbb{R}}[0, \pi]$ was investigated in [14], and for the case $\alpha = \pi$, $\beta \in (0, \pi)$ in [15], in original statement of a question, and the solution of these problems the authors reduced to analysis of inverse Sturm-Liouville problem with Dirichlet boundary conditions ($y(0) = 0$, $y(\pi) = 0$, which corresponds to $\alpha = \pi$, $\beta = 0$), which was in detail investigated in book [16]. We can say, that the problems, solved in [14, 15], are not the inverse problems by "spectral function", but these problems are deeply connected with the inverse problems by "spectral function", and [14, 15] play an important role in development of inverse problems. After, the authors of paper [17]

was studied the solution of inverse Sturm-Liouville problem $L(q, \pi, \beta)$ (i.e. $\alpha = \pi, \beta \in (0, \pi)$) with $q \in L^2_{\mathbb{R}}[0, \pi]$ in terms of eigenvalues and "norming constants" $\{\nu_n\}_{n=0}^{\infty}$ (see (1.3) in [17]), which they introduced for this case ². They proved that the condition (1.11) is necessary for norming constants $b_n(q, \pi, \beta)$ (see (1.5)). They also proved that the conditions (1.10) and (1.11) are necessary for norming constants of problem $L(q, \alpha, \beta)$, if $\sin \alpha \neq 0$ and $\sin \beta \neq 0$. Really they formulated these relations in terms of "norming constants" $\{\nu_n\}_{n=0}^{\infty}$, but it is easy to verify that these formulations are equivalent. Thus, Theorem 1.2 was proved, for the case $q \in L^2_{\mathbb{R}}[0, \pi]$, with different methods, in [17] and [13]. It is easy to see that these proofs remain the same for the case $q \in L^1_{\mathbb{R}}[0, \pi]$. It is also must be noted, that the relations (1.10) and (1.11) come from the paper [18] of Jodeit and Levitan.

Note that asymptotic behavior of $\{\mu_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ are standard conditions for the solvability of the inverse problem (see, e.g., [7, 8, 9, 12]). The conditions (1.8) and (1.9), which we add to the conditions (1.6) and (1.7), guarantee that α and β , which we construct during the solution of the inverse problem, are the same that we fixed in advance. At the same time Theorem 1.2 says that the conditions (1.8) and (1.9), which equivalent to (1.10) and (1.11) are necessary.

2. AUXILIARY RESULTS

Consider the function $a(x)$, defined as

$$(2.1) \quad a(x) = \sum_{n=0}^{\infty} \left(\frac{\cos \lambda_n x}{a_n} - \frac{\cos nx}{a_n^0} \right),$$

where $a_0^0 = \pi$, $a_n^0 = \frac{\pi}{2}$, for $n = 1, 2, \dots$

Lemma 2.1. *Let the sequences $\{\lambda_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ have the properties (1.6) and (1.7) correspondingly. Then $a \in AC(0, 2\pi)$.*

Proof. We set

$$(2.2) \quad \rho_n = \lambda_n - n = \frac{\omega}{n} + l_n = O\left(\frac{1}{n}\right).$$

The general term of the sum (2.1) can be rewritten as follows

²We should say, that this "norming constants" had been introduced before in paper [19], and corresponding uniqueness theorem (see theorem 2.3 in [17]) had been proved in [19] (see theorem 3).

$$\begin{aligned}
(2.3) \quad \frac{\cos \lambda_n x}{a_n} - \frac{\cos nx}{a_n^0} &= \frac{\cos \lambda_n x}{a_n} - \frac{\cos nx}{a_n} + \frac{\cos nx}{a_n} - \frac{\cos nx}{a_n^0} = \\
&= \frac{1}{a_n} (\cos \lambda_n x - \cos nx) + \left(\frac{1}{a_n} - \frac{1}{a_n^0} \right) \cos nx.
\end{aligned}$$

So we can rewrite the series (2.1) in the following form

$$(2.4) \quad a(x) = \sum_{n=0}^{\infty} \frac{1}{a_n} (\cos \lambda_n x - \cos nx) + \sum_{n=0}^{\infty} \left(\frac{1}{a_n} - \frac{1}{a_n^0} \right) \cos nx.$$

The difference $(\cos \lambda_n x - \cos nx)$ can be represented as follows

$$\begin{aligned}
(2.5) \quad \cos \lambda_n x - \cos nx &= \cos(n + \rho_n)x - \cos nx = \\
&= \cos nx \cos \rho_n x - \sin nx \sin \rho_n x - \cos nx = \\
&= -\cos nx(1 - \cos \rho_n x) - \sin nx \sin \rho_n x = \\
&= -2 \sin^2 \frac{\rho_n x}{2} \cos nx - \rho_n x \sin nx - (\sin \rho_n x - \rho_n x) \sin nx
\end{aligned}$$

and for the difference $\left(\frac{1}{a_n} - \frac{1}{a_n^0} \right)$ we have

$$(2.6) \quad \frac{1}{a_n} - \frac{1}{a_n^0} = \frac{1}{\frac{\pi}{2} + s_n} - \frac{1}{\frac{\pi}{2}} = -\frac{2}{\pi} \cdot \frac{s_n}{1 + \frac{\pi}{2} s_n} = -\frac{2}{\pi} \cdot s_n + q_n,$$

where $q_n = o\left(\frac{1}{n^2}\right)$. From the latter relation (2.6) it follows, that for sufficiently large n we have

$$(2.7) \quad \frac{1}{a_n} = \frac{1}{a_n^0} + o\left(\frac{1}{n}\right) = \frac{2}{\pi} + o\left(\frac{1}{n}\right).$$

And then, according to the relations (2.2) and (2.7), we obtain

$$\begin{aligned}
(2.8) \quad \frac{1}{a_n} (-\rho_n x) \sin nx &= \left[-\frac{2}{\pi} + o\left(\frac{1}{n}\right) \right] \left[\frac{\omega}{n} + l_n \right] x \sin nx = \\
&= -\frac{2}{\pi} \omega x \frac{\sin nx}{n} - \frac{2}{\pi} l_n x \sin nx + r_n \sin nx,
\end{aligned}$$

where $r_n = o\left(\frac{1}{n^2}\right)$.

Since $\rho_n = O\left(\frac{1}{n}\right)$ and $\sin y - y = O(n^3)$ for y close to zero, then

$$(2.9) \quad \sin \rho_n x - \rho_n x = O((\rho_n x)^3) = O\left(\frac{1}{n^3}\right),$$

$$(2.10) \quad \sin^2 \frac{\rho_n x}{2} = O\left(\left(\frac{\rho_n x}{2}\right)^2\right) = O\left(\frac{1}{n^2}\right).$$

Thus, taking into account the relations (2.5)–(2.10) we can rewrite the function (2.4) as follows

$$a(x) = a_1(x) + a_2(x),$$

where

$$\begin{aligned} a_1(x) &= -\frac{2\omega x}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n} - \frac{2x}{\pi} \sum_{n=1}^{\infty} l_n \sin nx - \frac{2}{\pi} \sum_{n=1}^{\infty} s_n \cos nx, \\ a_2(x) &= -\sum_{n=1}^{\infty} \frac{1}{a_n} (\sin \rho_n x - \rho_n x) \sin nx - \sum_{n=0}^{\infty} \frac{1}{a_n} \sin^2 \frac{\rho_n x}{2} \cos nx + \\ &\quad + \sum_{n=0}^{\infty} q_n \cos nx + \sum_{n=1}^{\infty} r_n \sin nx. \end{aligned}$$

Since the first sum $\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2}$, for $x \in (0, 2\pi)$, then, in particular, it belongs to $AC(0, 2\pi)$, and the second sum $\sum_{n=1}^{\infty} l_n \sin nx$ also belongs to $AC(0, 2\pi)$ according to the condition (1.6d) of the lemma. In its turn the sum $\sum_{n=1}^{\infty} s_n \cos nx$ belongs to $AC(0, 2\pi)$ according to the condition (1.7d) of the lemma. The other four sums of $a_2(x)$ converge absolutely and uniformly on $[0, 2\pi]$ and are continuous differentiable functions, and hence belong to $AC(0, 2\pi)$.

These complete the proof. \square

Lemma 2.2. *Let the sequences $\{\lambda_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ have the properties (1.6) and (1.7), (1.8) correspondingly. Then the function F , defined in triangle $0 \leq t \leq x \leq \pi$ by formula*

$$(2.11) \quad F(x, t) = \sum_{n=0}^{\infty} \left(\frac{\cos \lambda_n x \cos \lambda_n t}{a_n} - \frac{\cos nx \cos nt}{a_n^0} \right),$$

is absolutely continuous function with respect to each variable and function

$$f(x) := \frac{d}{dx} F(x, x),$$

is summable on $(0, \pi)$, i.e. $f \in L^1(0, \pi)$.

Proof. It is easy to see that

$$(2.12) \quad F(x, t) = \frac{1}{2}[a(x+t) + a(x-t)].$$

Since, $a \in AC(0, 2\pi)$, then we can infer that the function $F(x, t)$ with respect to both of the variables has the same smoothness as $a(x)$. For the functions $F(x, x)$ we have

$$(2.13) \quad F(x, x) = \frac{1}{2}[a(2x) + a(0)].$$

According to (1.8) and (2.1)

$$a(0) = \sum_{n=0}^{\infty} \left(\frac{1}{a_n} - \frac{1}{a_n^0} \right) = \cot \alpha,$$

for $\alpha \in (0, \pi)$. Hence $a(0)$ has a sense and, therefore, $F(x, x)$ too. Besides this

$$(2.14) \quad \frac{d}{dx}F(x, x) = \frac{1}{2} \frac{d}{dx}a(2x),$$

and since $a \in AC(0, 2\pi)$, then the function $a(2x)$ belongs to $AC(0, \pi)$, and its derivative belongs to $L_{\mathbb{R}}^1(0, \pi)$, i.e. the function $\frac{d}{dx}F(x, x)$ belongs to $L_{\mathbb{R}}^1(0, \pi)$. \square

3. PROOF OF THE THEOREM 1.1

The proof of necessity. If $\{\lambda_n^2\}_{n=0}^{\infty}$ are the eigenvalues and $\{a_n\}_{n=0}^{\infty}$ are the norming constants of the problem $L(q, \alpha, \beta)$, then for $\mu_n = \lambda_n^2$ the asymptotics (1.6) was proved in [20], and for a_n the asymptotics (1.7) was proved in [21]. The necessity of connections (1.8) and (1.9) was proved in [13] for $q \in L_{\mathbb{R}}^2[0, \pi]$, and the same proof is true for the case $q \in L_{\mathbb{R}}^1[0, \pi]$.

The proof of sufficiency. In [8] there is a proof of such assertion:

Theorem 3.1 ([8]). *For real numbers $\{\lambda_n^2\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ to be the spectral data for a certain boundary-value problem $L(q, \alpha, \beta)$ with $q \in L_{\mathbb{R}}^1[0, \pi]$, $(\alpha, \beta \in (0, \pi))$, it is necessary and sufficient that relations (1.6a)–(1.6b) and (1.7a)–(1.7b) hold, and the function $F(\cdot, \cdot)$ has partial derivatives, which are summable with respect to each variable.*

Thus, if we have a real sequence $\{\mu_n\}_{n=0}^{\infty} = \{\lambda_n^2\}_{n=0}^{\infty}$, which has the asymptotic representation (1.6a)–(1.6b) and a positive sequence $\{a_n\}_{n=0}^{\infty}$, which has the asymptotic representation (1.7a)–(1.7b), then, according to the Theorem 3.1, there exist a function $q \in L_{\mathbb{R}}^1[0, \pi]$ and

some constants $\tilde{\alpha}, \tilde{\beta} \in (0, \pi)$ such that λ_n^2 , $n = 0, 1, 2, \dots$, are the eigenvalues and a_n , $n = 0, 1, 2, \dots$, are norming constants of a Sturm-Liouville problem $L(q, \tilde{\alpha}, \tilde{\beta})$.

The function $q(x)$ and constants $\tilde{\alpha}, \tilde{\beta}$ are obtained on the way of solving the inverse problem by Gelfand-Levitan method. The algorithm of that method is as follows:

First we define the function $F(x, t)$ by formula

$$(3.1) \quad F(x, t) = \sum_{n=0}^{\infty} \left(\frac{\cos \lambda_n x \cos \lambda_n t}{a_n} - \frac{\cos nx \cos nt}{a_n^0} \right).$$

Note that this function is defined by $\{\lambda_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ uniquely. Then we solve Gelfand-Levitan integral equation [4, 7, 8, 12]

$$(3.2) \quad G(x, t) + F(x, t) + \int_0^x G(x, s)F(s, t)ds = 0, \quad 0 \leq t \leq x,$$

where $G(x, \cdot)$ is unknown function. Find function $G(x, t)$, with the help of which we construct a function

$$(3.3) \quad \varphi(x, \lambda^2) = \cos \lambda x + \int_0^x G(x, t) \cos \lambda t dt,$$

which is defined for all $\lambda \in \mathbb{C}$. It is proved (see [8]) that

$$(3.4) \quad -\varphi''(x, \lambda^2) + \left(2 \frac{d}{dx} G(x, x) \right) \varphi(x, \lambda^2) = \lambda^2 \varphi(x, \lambda^2),$$

almost everywhere on $(0, \pi)$, and

$$\begin{aligned} \varphi(0, \lambda^2) &= 1, \\ \varphi'(0, \lambda^2) &= G(0, 0). \end{aligned}$$

If we denote

$$(3.5) \quad G(0, 0) = -\cot \tilde{\alpha},$$

then the solution (3.3) of equation (3.4) will satisfy the boundary condition (1.2)

$$\varphi(0, \lambda^2) \cot \tilde{\alpha} + \varphi'(0, \lambda^2) = 0$$

for all $\lambda \in \mathbb{C}$. Since from (3.2) follows that $G(0, 0) = -F(0, 0)$ and from (3.1) follows that

$$F(0, 0) = \sum_{n=0}^{\infty} \left(\frac{1}{a_n} - \frac{1}{a_n^0} \right), \text{ hence we get}$$

$$(3.6) \quad \sum_{n=0}^{\infty} \left(\frac{1}{a_n} - \frac{1}{a_n^0} \right) = \cot \tilde{\alpha}.$$

From the relation (3.6) and our condition (1.8) on the sequence $\{a_n\}_{n=0}^{\infty}$ we find that $\tilde{\alpha} = \alpha$.

It is also proved (see, e.g., [8]) that the expression

$$\frac{\varphi'_n(\pi)}{\varphi_n(\pi)} = \frac{\varphi'(\pi, \lambda_n^2)}{\varphi(\pi, \lambda_n^2)}$$

is a constant (i.e. does not depend on n), which we will denote by $-\cot \tilde{\beta}$. Thus the functions $\varphi(x, \lambda_n^2), n = 0, 1, 2, \dots$, are the eigenfunctions of a problem $L(q, \tilde{\alpha}, \tilde{\beta})$, where $q(x) = 2 \frac{d}{dx} G(x, x)$, $\tilde{\alpha}$ is in advance given α and we should have $\tilde{\beta}$ equals β . We know from the Theorem 1.2, that for problem $L(q, \alpha, \tilde{\beta})$ it holds

$$\frac{1}{b_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{b_n} - \frac{2}{\pi} \right) = -\cot \tilde{\beta}.$$

Thus, if we obtain condition (1.11), then we guarantee that $\tilde{\beta} = \beta$. But (1.11) deals with the norming constants b_n , which are not independent. We have shown in the paper [13] that we can represent b_n by a_n and $\{\mu_k\}_{k=0}^{\infty}$ by formulae (1.14), (1.15). Therefore, instead of (1.11), we obtain the condition in the form (1.9).

This completes the proof.

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