Mathematical Inverse Problems, Vol. 5, No. 1 (2018), 20-34 ISSN 2381-9634

RECOVERING THE SOURCE FUNCTION IN THE ADVECTION DIFFUSION EQUATION

JOSE ALCALA BURGOS[1] AND HALA AH SHEHADEH[2,*]

[1]Centro de Investigacion en Matematicas (CIMAT)-Merida
Kueski Ontario 1090, Italia Providencia Guadalajara, Jalisco, Mexico. CP. 44648
[2]Department of Mathematics and Statistics, James Madison University, Harrisonburg VA 22801
[*]Corresponding author: alhajjhy@jmu.edu

Received Sep 30, 2017

Abstract. We discuss the cases when the source $f(x)$ and the velocity ψ in the advection diffusion equation $u_t(x,t) = \Delta u(x,t) + \psi \cdot \nabla u(x,t) + f(x)$ can be uniquely recovered using *one future time measurement* $u(x,T)$ at all locations $x \in \mathbb{R}^n$. We consider only constant velocity functions ψ. This problem is important in threat detection applications, where we think of $f(x)$ as a source of a contaminant propagating in an urban area. We also present numerical simulations.

Key words and phrases. Inverse source problem, advection diffusion equation, Fourier transform, Paley- Wiener theorem, projection slice theorem, adjoint method.

1. Introduction

The propagation of contaminants in a medium can be modeled using the advection diffusion equation,

$$u_t(x,t) = \Delta u(x,t) + \psi \cdot \nabla u(x,t) + f(x), \tag{1}$$

with initial conditions $u(x,0) = 0$, where $u(x,t)$ is the contaminant density at the position $x \in \mathbb{R}^d$ and time $t \in [0,\infty)$, ψ is the *constant* velocity of the medium, and $f(x)$ is the contaminant source intensity at the position x. In applications, $u(x,t)$, ψ and $f(x)$ are all unknown except for few measurements at given positions and times (the time T at which a certain measurement is taken could also be unknown, in the cases where it is not clear when the contamination started). The inverse problem is to recover $u(x,t)$, ψ and $f(x)$ from the known measurements.

A natural question to ask is *how many measurements* $u(\cdot, T_i)$ *are required to recover* ψ *and* $f(x)$ *(and consequently* $u(x,t)$*) uniquely*? In this paper we prove that *only a one time measurement* $u(\cdot, T)$ is enough to solve the inverse problem, given that the source function $f(x)$ is in $L^2(\mathbb{R}^d)$ and has *compact support*.

©2018 Mathematical Inverse Problems

In the literature [7], it is common to use two or more measurements. Methods using Carleman estimates require knowing the solution $u(x,t)$ at a continuous time interval [7]. We argue that is unnecessary for this case. Other analytical and numerical work have been done on recovering special source functions (see [3, 4, 5, 6] for example and the references therein).

In section 2 we establish our uniqueness result and its proof. We start with the one dimensional case, then generalize to the d-dimensional setting. Section 3 discusses the steady state case.

In section 4 we formulate the numerical problem and present the results of the numerical simulations. Our conclusions are discussed in section 6. For the sake of self inclusion, the theorems from the literature that are necessary to establish our proofs are stated in Appendix A.

In all of the following, the source function f is assumed to be nonzero.

2. Constant Velocity ψ

2.1. **One dimensional case.** For simplicity, we start with the one dimensional case ($x \in \mathbb{R}$), since it captures most of the ideas. The arguments for the d-dimensional case are analogous and will be presented in section 2.2. We consider the problem in Fourier space where for a function $h(x)$ in $L^2(\mathbb{R})$:

$$\hat{h}(\xi) = \int_{\mathbb{R}} h(x)e^{-i\xi x}dx. \tag{2}$$

Taking the Fourier transform of (1) we obtain the ODE

$$\hat{u}_t(\xi,t) + (\xi^2 - i\xi\psi)\hat{u}(\xi,t) - \hat{f}(\xi) = 0\,, \tag{3}$$

and solving for $\hat{u}(\xi,t)$ using Duhamel's principle, we have

$$\hat{u}(\xi,t) = \hat{f}(\xi)\int_0^t e^{-(\xi^2 - i\xi\cdot\psi)(t-s)}ds, \tag{4}$$

where $\xi \in \mathbb{R}, t \in [0,\infty)$. Solving the integral in the above expression, we get

$$\hat{u}(\xi,t) = \begin{cases} \dfrac{\hat{f}(\xi)}{\xi^2 - i\psi\cdot\xi}\left(1 - e^{-\left(\xi^2 - i\psi\cdot\xi\right)t}\right) \text{ if } \xi \neq 0, \\ t\hat{f}(0) \text{ if } \xi = 0. \end{cases} \tag{5}$$

Assuming that the measurement $u(x,T)$ (and hence $\hat{u}(\xi,T)$) is known at a fixed time $T \in (0,\infty)$, we address *uniqueness*: are there different source functions $f_1(x)$, $f_2(x)$ and velocity constants ψ_1, ψ_2 such that the corresponding solutions $u_1(x,T)$, $u_2(x,T)$ of (1) satisfy $u_1(x,T) = u_2(x,T)$ for all $x \in \mathbb{R}$? The answer is *no* and this is argued in Theorem 1 below, which gives the uniqueness result in the one dimensional case. But first, we have the following remark.

Remark 1. (1) *If $\psi_1 = \psi_2$, then it is obvious from (4) that $\hat{f}_1(\xi) = \hat{f}_2(\xi)$ and so are $f_1(x)$ and $f_2(x)$.*

(2) *If $f_1(x) = f_2(x)$ then $\psi_1 = \psi_2$ as well. We can see this if we define*

$$a_j = \xi^2 - i\psi_j \cdot \xi, \;\; j = 1, 2, \tag{6}$$

then by referring to (5) and since $\hat{u}_1(\xi, T) = \hat{u}_2(\xi, T)$ we have,

$$\hat{f}_1(\xi)\frac{-1 + e^{-a_1 T}}{-a_1} = \hat{f}_2(\xi)\frac{-1 + e^{-a_2 T}}{-a_2}, \;\; \xi \in \mathbb{R}. \tag{7}$$

Since $\hat{f}_1(\xi) = \hat{f}_2(\xi)$, and Taylor expanding the exponentials in (7), we have,

$$T - \frac{a_1 T^2}{2!} + \frac{a_1^2 T^3}{3!} - \ldots = T - \frac{a_2 T^2}{2!} + \frac{a_2^2 T^3}{3!} - \ldots. \tag{8}$$

Plugging a_j from (6) in (8) and comparing the coefficients of ξ to first order, we arrive at $\psi_1 = \psi_2$.

So the uniqueness of f implies the uniqueness of ψ and vice versa.

Theorem 1. *Let $u(x,t)$ be a solution of the advection diffusion equation,*

$$u_t(x,t) = \Delta u(x,t) + \psi \cdot \nabla u(x,t) + f(x), \tag{9}$$

with $u(x,0) = 0$, where $f(x) \in L^2(\mathbb{R})$ is compactly supported and $\psi \in \mathbb{R}$ is a constant.

Let $T \in (0,\infty)$ and $g(x) \in C^2(\mathbb{R})$ be such that $u(x,T) = g(x)$. Then $f(x)$ and ψ are unique, in the sense that: If we have compactly supported functions $f_1, f_2 \in L^2(\mathbb{R})$, constants $\psi_1, \psi_2 \in \mathbb{R}$, and corresponding solutions $u_1(x,t)$ and $u_2(x,t)$ of (9) with $u_1(x,0) = u_2(x,0) = 0$ and $u_1(x,T) = u_2(x,T) = g(x)$, then $\psi_1 = \psi_2$ and $f_1(x) = f_2(x)$ for all $x \in \mathbb{R}$.

Proof. We argue by contradiction.

Suppose we have f_1, f_2, ψ_1, ψ_2, $u_1(x,t)$, and $u_2(x,t)$ that satisfy equation (9) with $u_1(x,0) = u_2(x,t) = 0$ and $u_1(x,T) = u_2(x,T) = g(x)$. Suppose also that $\psi_1 \neq \psi_2$. Then by (5) and the final time condition at $t = T$, $\hat{f}_1(\xi)$ and $\hat{f}_2(\xi)$ must satisfy,

$$\hat{f}_2(\xi) = \hat{f}_1(\xi)\frac{a_2}{a_1}\frac{\left(-1 + e^{-a_1 T}\right)}{\left(-1 + e^{-a_2 T}\right)}, \tag{10}$$

where a_1 and a_2 are given by (6). The main idea of the proof is as follows:

(a) $\hat{f}_2(\xi)$ given by (10) is well defined for all $\xi \in \mathbb{R}$,

(b) $\hat{f}_2(\xi)$ given by (10) is the Fourier transform of a compactly supported function $f_2(x) \in L^2(\mathbb{R})$,

(c) The extensions $\hat{f}_1(z)$ and $\hat{f}_2(z)$ of $\hat{f}_1(\xi)$ and $\hat{f}_2(\xi)$ respectively to the complex plane $\mathbb{C}$ are entire functions (since they are extensions of the Fourier transforms of compactly supported functions),

while at the same time,

(d) $\hat{f}_1(\xi)$ and $\hat{f}_2(\xi)$ have at most $O(r)$ zeros inside a ball of radius r in $\mathbb{C}$,

(e) $\left(-1+e^{a_2 T}\right)$ has $O(r^2)$ zeros inside a ball of radius r, all of which are distinct from the zeros of $\left(-1+e^{a_1 T}\right)$.

Hence, the zeros of the numerator cannot cancel those of the denominator in (10), contradicting the fact that $\hat{f}_2(\xi)$ is entire.

We are now left with the task justifying the above statements (a through e). (b) and (c) are obvious.

To prove (a), (d) and (e), Theorem 4 (Appendix A) due to Paley and Wiener is essential, which says that the extension to $\mathbb{C}$ of the Fourier transform of a compactly supported L^2-function can grow at most exponentially fast:

From (4) and (5), (10) can be written as

$$\hat{f}_2(\xi) = \hat{f}_1(\xi) \frac{\int_0^t e^{-\left(\xi^2 - i\psi_1 \cdot \xi\right)(t-s)} ds}{\int_0^t e^{-\left(\xi^2 - i\psi_2 \cdot \xi\right)(t-s)} ds}, \ \xi \in \mathbb{R}. \tag{11}$$

Define a new function $h(z)$ to be the extension of $\hat{f}_2(\xi)$ given in (11) to the entire complex plane:

$$\begin{aligned} h(z) &= \hat{f}_1(z) \frac{\int_0^t e^{-\left(z^2 - i\psi_1 z\right)(t-s)} ds}{\int_0^t e^{-\left(z^2 - i\psi_2 z\right)(t-s)} ds} \\ &= \hat{f}_1(z) \frac{\left(z^2 - i\psi_2 z\right)\left(1 - e^{-\left(z^2 - i\psi_1 z\right)T}\right)}{\left(z^2 - i\psi_1 z\right)\left(1 - e^{-\left(z^2 - i\psi_2 z\right)T}\right)}, \ z \in \mathbb{C}. \end{aligned} \tag{12}$$

Obviously, the $1 - e^{-\left(z^2 - i\psi_2 z\right)T}$ part of the denominator in (12) is entire. A simple calculation shows that its zeros satisfy $z^2 - i\psi_2 z - 2\pi i n/T = 0, n \in \mathbb{Z}$, and that these do not cancel with the zeros of $1 - e^{-\left(z^2 - i\psi_1 z\right)T}$ in the numerator. Moreover, these zeros are of order $O(r^2)$ in a ball $|z| < r$ (since we can explicitly calculate $z \sim \sqrt{n}$, so if $|z| < r$, then $n \sim r^2$).

The zeros of $\hat{f}_1(z)$ do not cancel those of the denominator in (12) either. This can be seen through the following argument: Since $\hat{f}_1(\xi)$ is the Fourier transform of a compactly supported function $f_1(x) \in L^2(\mathbb{R})$, then using Theorem 4 we have

$$|\hat{f}_1(z)| \leq C \exp(A|z|), \tag{13}$$

for some constants A and C. If $n(r)$ is the number of zeros of $\hat{f}_1(z)$ inside a circle $|z| = r$, then using Jensen's formula (see (2) on page 309 in [1] for instance), $n(r)$ satisfies

$$n(r)\log 2 \leq \log\left(\sup_{\theta}\left|\hat{f}_1\left(2re^{i\theta}\right)\right|\right). \tag{14}$$

This with (13) implies that

$$n(r) \leq \frac{2CAr}{\log 2}. \tag{15}$$

Hence, the number of zeros of $\hat{f}_1(z)$ in a ball of radius r can at most grow linearly with r.

Counting the number of zeros in the numerator and denominator of $h(z)$ in (12), we deduce that $h(z)$ is not an entire function, in other words, $\hat{f}_2(\xi)$ cannot be extended to an entire function in $\mathbb{C}$. Therefore, $\hat{f}_2(\xi)$ given by (10) cannot be the Fourier transform of a compactly supported function in $L^2(\mathbb{R})$ (again due to Theorem 4), leading to our contradiction.

□

2.2. **d-dimensional case.** The extension of the above uniqueness result to the $\mathbb{R}^d$ case is natural and we will do it using the *projection slice theorem* for Fourier transforms, which is discussed in Appendix A (Theorem 5).

We start by writing equation (1) in Fourier space,

$$\hat{u}_t = -|\xi|^2\hat{u} + i\psi\cdot\xi\hat{u} + \hat{f}, \xi \in \mathbb{R}^d, t \in [0,\infty). \tag{16}$$

The solution of (16) is given by

$$\begin{aligned}\hat{u}(\xi,t) &= \hat{f}(\xi)\int_0^t e^{-\left(|\xi|^2 - i\psi\cdot\xi\right)(t-s)}ds \\ &= \begin{cases}\dfrac{\hat{f}(\xi)}{|\xi|^2 - i\psi\cdot\xi}\left(1 - e^{-\left(|\xi|^2-i\psi\cdot\xi\right)t}\right) \text{ if } \xi \neq 0, \\ t\hat{f}(\xi) \text{ if } \xi = 0.\end{cases}\end{aligned} \tag{17}$$

Theorem 2. *Let $u(x,t)$ be a solution of the advection diffusion equation with $u(x,0) = 0$, where $f(x) \in L^2(\mathbb{R}^n)$ is compactly supported and $\psi \in \mathbb{R}^n$ is a constant. Let $T \in (0,\infty)$ and $g(x) \in C^2(\mathbb{R}^n)$ be such that $u(x,T) = g(x)$. Then $f(x)$ and ψ are unique, in the sense that: If we have compactly supported functions $f_1, f_2 \in L^2(\mathbb{R}^n)$, constants $\psi_1, \psi_2 \in \mathbb{R}^n$, and corresponding solutions $u_1(x,t)$ and $u_2(x,t)$ of (9) with $u_1(x,0) = u_2(x,0) = 0$ and $u_1(x,T) = u_2(x,T) = g(x)$, then $\psi_1 = \psi_2$ and $f_1(x) = f_2(x)$ for all $x \in \mathbb{R}^n$.*

Proof. If $\psi_1 = \psi_2$, we obtain $f_1 = f_2$ from (17) and the uniqueness of the Fourier transform. So it is enough to prove that $\psi_1 = \psi_2$. From equation (17) we have

$$\hat{f}_2(\xi) = \hat{f}_1(\xi)\frac{\int_0^t e^{-\left(|\xi|^2-i\psi_1\cdot\xi\right)(t-s)}ds}{\int_0^t e^{-(|\xi|^2-i\psi_2\cdot\xi)(t-s)}ds} \tag{18}$$

Since $f_1 \in L^2(\mathbb{R}^n)$ is nonzero, its Fourier transform is nonzero and by the Paley-Wiener theorem it can be extended to an entire function in $\mathbb{C}^n$. It follows from the continuity of $\widehat{f_1}$ that there is $\xi^1 \in \mathbb{R}^n$ and a ball of radius r around ξ^1, denoted by $B_r(\xi^1)$, such that $\widehat{f_1}$ is non zero for all elements in the ball. Denote by U_r the set of unit vectors $\nu = \frac{\xi}{|\xi|}$ where $\xi \neq 0$ belongs to $B_r(\xi^1)$. Evaluating (18) for a fixed $\nu \in B_r(\xi^1)$ and $\eta \in \mathbb{R}^n$ we obtain

$$\hat{f}_2(\eta\nu) = \hat{f}_1(\eta\nu) \frac{\int_0^t e^{-\left(\eta^2 - i\eta\psi^1\cdot\nu\right)(t-s)} ds}{\int_0^t e^{-(\eta^2 - i\eta\psi^2\cdot\nu)(t-s)} ds}, \ \xi_1 \in \mathbb{R}. \tag{19}$$

This will be equivalent to equation (10), used to determine the uniqueness of $\psi \cdot \nu$ and g if $\hat{f}_1(\eta\nu) = \widehat{g_1}(\eta)$ and $\hat{f}_2(\eta\nu) = \widehat{g_2}(\eta)$ for some scalar compactly supported functions g_1 and g_2 in $L^2(\mathbb{R})$. We use Theorem 5 to prove that this is the case. Define

$$g_i(l) = \int_{x\cdot\nu = l} f_i(x) dA = P_\nu[f_i](l), \qquad i = 1, 2 \tag{20}$$

and by the slice theorem 5,

$$\begin{aligned} \widehat{g_i}(\eta) &= \widehat{P_\nu[f_i]}(\eta) \\ &= S_\nu[\hat{f}_i](\eta) \\ &= \hat{f}_i(\eta\nu). \end{aligned} \tag{21}$$

Since $f_1(x)$ and $f_2(x)$ are compactly supported then so are g_1 and g_2. Moreover, g_1 and g_2 belong to $L^2(\mathbb{R})$. This can be seen using Fubini's Theorem, and the fact that $f_1, f_2 \in L^2(\mathbb{R}^n)$:

Let χ_f be the characteristic function of the support of f, supp$f$, and let supp$f \subset [-a, a]^d$ for some constant a. We have

$$\begin{aligned} \int_{\mathbb{R}} (g(l))^2 \, dl &= \int_{\mathbb{R}} \left(\int_{x\cdot\nu=l} f(x)\chi_f dA \right)^2 dl \\ &\leq \int_{\mathbb{R}} \int_{x\cdot\nu=l} f^2 dA \int_{x\cdot\nu=l} \chi_f dA dl \\ &\leq (2a)^{d-1} \int_{\mathbb{R}} \int_{x\cdot\nu=l} f^2 dA dl \\ &= (2a)^{d-1} \int_{\mathbb{R}^n} f^2 dx < \infty. \end{aligned} \tag{22}$$

The last equality in (22) uses Fubini's Theorem.

Therefore, by our one dimensional analysis, $(\psi_1 - \psi_2) \cdot \nu = 0$ and $g_1 = g_2$. The previous arguments holds for all ν in an open set of the unit sphere in $\mathbb{R}^n$, so we can conclude that $\psi_1 = \psi_2$. This finishes the proof of the theorem. $\square$

Remark 2. *It is well known that Paley-Wiener theorem (Theorem 4) is valid in d-dimensions so the proof of Theorem 1 can easily be extended to the d-dimensional case (without necessarily going through the projection slice theorem). However, the projection slice theorem (Theorem 5) is very useful in inverse problems applications so we opted to use it.*

3. Steady State

The steady state solution $v(x)$ satisfies

$$\Delta v(x) + \epsilon(x) \cdot \nabla v(x) + f(x) = 0, \;\; x \in \mathbb{R}^n \tag{23}$$

The following theorem asserts that in this case, there is *no uniqueness* unless we are among characteristic functions.

Theorem 3. *Given* $(\epsilon_1(x), f_1(x))$ *and* $v(x)$ *such that*

$$\Delta v(x) + \epsilon_1(x) \cdot \nabla v(x) + f_1(x) = 0, \;\; x \in \mathbb{R}^n, \tag{24}$$

then we can find $(\epsilon_2(x), f_2(x))$ *that also satisfy*

$$\Delta v(x) + \epsilon_2(x) \cdot \nabla v(x) + f_2(x) = 0, \;\; x \in \mathbb{R}^n \tag{25}$$

(with the same $v(x)$*).*

Proof. From (23) we have $f(x) = -(\Delta v(x) + \epsilon(x) \cdot \nabla v(x))$. Given $v(x)$, it is known via maximum principle arguments that the uniqueness of $\epsilon(x)$ implies that of $f(x)$. If $\epsilon(x)$ is non-unique, and there are two distinct sources $f_1(x)$ and $f_2(x)$, then $f_1(x) - f_2(x) = (\epsilon_2(x) - \epsilon_1(x)) \cdot \nabla v(x)$. Hence, distinct velocities imply distinct sources unless $(\epsilon_2(x) - \epsilon_1(x))$ is orthogonal to $\nabla v(x)$. Therefore, in general, the sources are non-unique: Given $v(x)$, $f_1(x)$ and $\epsilon_1(x)$, choose $\epsilon_2(x)$ such that $(\epsilon_2(x) - \epsilon_1(x))$ is not orthogonal to $\nabla v(x)$, and $f_2(x) = -(\epsilon_2(x) - \epsilon_1(x)) \cdot \nabla v(x) + f_1(x)$. Then $(\epsilon_1(x), f_1(x))$ and $(\epsilon_2(x), f_2(x))$ will produce the same measurement $v(x)$ (for infinitely large time). ☐

The above argument shows that we *cannot* recover $f(x)$ and ψ uniquely from the steady state, even when ψ is *constant*. The reason is that as $t \to \infty$, the zeros of the denominator in (10) disappear, and hence the proof of Theorem 1 does not work.

4. Numerical Method and Simulations

The goal is to numerically recover a compactly supported L^2 source function f using one future time measurement $u_d(\cdot) = u(\cdot, T)$ of the solution of the advection diffusion equation (1). We write the following minimization problem:

$$\min_{\substack{u,\psi,f \\ g(u,\psi,f)=0}} \;\; F(u, \psi, f), \tag{26}$$

where

$$F(u,\psi,f) = \frac{1}{2}\int_{\mathbb{R}} (u(x,T) - u_d(x))^2 \mathrm{d}x\ , \tag{27}$$

and

$$g(u,\psi,f) := \begin{pmatrix} u_t - u_{xx} - \psi u_x - f(x) \\ u(x,0) \end{pmatrix} . \tag{28}$$

We perform the minimization of the functional F by calculating its derivatives with respect to f and ψ using *the adjoint method* discussed in [2]. The idea is similar to Lagrange multipliers method for minimizing a real valued function on $\mathbb{R}^n$ subject to equality constraints, but in a functional analytical setting. We end up having to solve the following *adjoint equation* for the *multiplier* $\lambda(x,t)$:

$$\begin{cases} \lambda_t + \lambda_{xx} - \psi\lambda_x = 0, \\ \lambda(x,T) = u(x,T) - u_d, \\ \lambda(x,0) = \lambda_0(x). \end{cases} \tag{29}$$

Now we explain how to derive (29). Following [2], we introduce the notation:

$$\left\langle \begin{pmatrix} h_1(x,t) \\ s_1(x) \end{pmatrix}, \begin{pmatrix} h_2(x,t) \\ s_2(x) \end{pmatrix} \right\rangle_{L^2} = \int_0^T \int_{\mathbb{R}} h_1(x,t)h_2(x,t)\mathrm{d}x\ \mathrm{d}t + \int_{\mathbb{R}} s_1(x)s_2(x)\mathrm{d}x\ .$$

Consider the *relaxed* minimization problem (unconstrained)

$$\min_{u,\psi,f} F(u,\psi,f) + \left\langle \begin{pmatrix} \lambda(x,t) \\ \lambda_0(x) \end{pmatrix}, g(u,\psi,f) \right\rangle_{L^2} . \tag{30}$$

We now set to zero the derivatives with respect to u, f and ψ of the objective functional in the above expression (30). A minimizer is attained when these derivatives are equal to zero.

(1) Derivative with respect to u:

$$F_u\dot{u} + \left\langle \begin{pmatrix} \lambda(x,t) \\ \lambda_0(x) \end{pmatrix}, g_u\dot{u} \right\rangle_{L^2} = 0, \tag{31}$$

with

$$F_u\dot{u} = \int_{\mathbb{R}} (u(x,T) - u_d(x))\dot{u}(x,T)dx, \tag{32}$$

and

$$g_u\dot{u} = \begin{pmatrix} \dot{u}_t - \dot{u}_{xx} - \psi\dot{u}_x \\ \dot{u}(x,0) \end{pmatrix}. \tag{33}$$

Integration by parts in (31) implies (29). Hence, setting the derivative with respect to u to zero leads to the adjoint problem (29) which allows us to solve for the multiplier $\lambda(x,t)$.

(2) Derivative with respect to f:

$$F_f \dot{f} + \left\langle \begin{pmatrix} \lambda(x,t) \\ \lambda_0(x) \end{pmatrix}, g_f \dot{f} \right\rangle_{L^2} = 0, \tag{34}$$

implying

$$F_f \dot{f} = -\left\langle \begin{pmatrix} \lambda(x,t) \\ \lambda_0(x) \end{pmatrix}, \begin{pmatrix} -\dot{f} \\ 0 \end{pmatrix} \right\rangle_{L^2}. \tag{35}$$

(3) Derivative with respect to ψ: A similar calculation leads to

$$F_\psi \dot{\psi} = -\left\langle \begin{pmatrix} \lambda(x,t) \\ \lambda_0(x) \end{pmatrix}, \begin{pmatrix} -u_x \dot{\psi} \\ 0 \end{pmatrix} \right\rangle_{L^2}. \tag{36}$$

Therefore, given ψ and f (initializing the numerical method), the above procedure allows us to obtain the derivatives of $F(u,\psi,f)$ with respect to ψ and f in three steps:

- Set $g = 0$ in (28) then calculate $u(x,t)$ (solve for $u(x,t)$ in the advection diffusion equation).
- Use (29) to calculate $\lambda(x,t)$ (solve the adjoint problem using $u(x,T)$ and the data $u_d(x)$ to find $\lambda(x,t)$).
- Plug $u(x,t)$ and $\lambda(x,t)$ in equations (35) and (36), then

$$F_f = \int_0^T \lambda(x,t)dt, \text{ and } F_\psi = \int_0^T \int_{\mathbb{R}} \lambda(x,t) u_x(x,t) dx dt \tag{37}$$

Now we can minimize F over ψ and f with a simple gradient descent:

$$f_{n+1} = f_n - \alpha F_f, \quad \psi_{n+1} = \psi_n - \beta F_\psi \tag{38}$$

To numerically compute the above derivatives, we need to discretize equations (28), (29), (35) and (36). The first two equations are parabolic with constant coefficients and time independent source term, so we can solve the equation exactly in Fourier space. The only numerical error comes from the quadrature rule used in the approximation of the integral in the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} \mathrm{d}x \,, \tag{39}$$

and in its inverse. Our quadrature rule depends on a radius parameter R and the even number of uniform subintervals N in the subdivision of $[-R/2, R/2]$ with elements of size

R/N. We use the right endpoint $x_k = -R/2 + k\frac{R}{N}$ on each subinterval and our approximation is

(40)
$$\overline{f}(\xi) = \sum_{k=1}^{N} f(x_k) e^{-i\xi x_k} \frac{R}{N}\,.$$

On the uniform grid $\xi_m = -N/2 + (m-1)$, we obtain

(41)
$$\overline{f}_m = \sum_{k=1}^{N} f(x_k) e^{-i\xi_m x_k} \frac{R}{N}\,.$$

We compute the sum using the non-uniform Fast Fourier Transform (NUFFT) [8, 9]. This is not strictly necessary, since we are evaluating the function f and its Fourier transform on uniform grids. We could have done the calculation with the Fast Fourier Transform (FFT) algorithm, but the interface in Python for the NUFFT [10] was easier to use and has similar performance.

To numerically collect the data $u_d(x) = u(x,T)$: Solve advection diffusion equation with the true source $f(x)$, measure solution $u(x,T)$ at time T then add noise.

We apply the above numerical method to an example where the source function $f(x)$ is given by a two peaked signal (blue curve in Figure 1). The red curve is the initial guess f_0. The blue curve in Figure 2 shows the data $u_d(x) = u(x,T) + \text{noise}$, where noise has standard deviation equal to 0.02. Figure 3 shows that the numerical method correctly recovers the source function, and Figure 4 shows that the method converged to a minimizer ($F_f = 0$). We initialize ψ with the value we want to recover, this is $\psi_0 = 900$. After the first iteration, we obtain $\psi_1 = 904.815857937$ because the source is very far from the one we want to recover. Further steps take it closer to the value we want to recover, with the final value being $\psi_{1000} = 900.796125067$.

The code (in Python) can be found at [11] and [12].

5. Discussion

To summarize our results, we have proved that:

(1) In the case the velocity ψ is constant, we can recover $f(x)$ and ψ in (1) uniquely using one measurement, given that $f(x)$ is a compactly supported L^2 function. In this case, uniqueness of $f(x)$ is equivalent to the uniqueness of ψ.
(2) In the steady state case, there is *no uniqueness* unless we are among characteristic functions.

We also describe a numerical method to recover compactly supported source functions and provide simulations demonstrating the main result (Theorem 1).

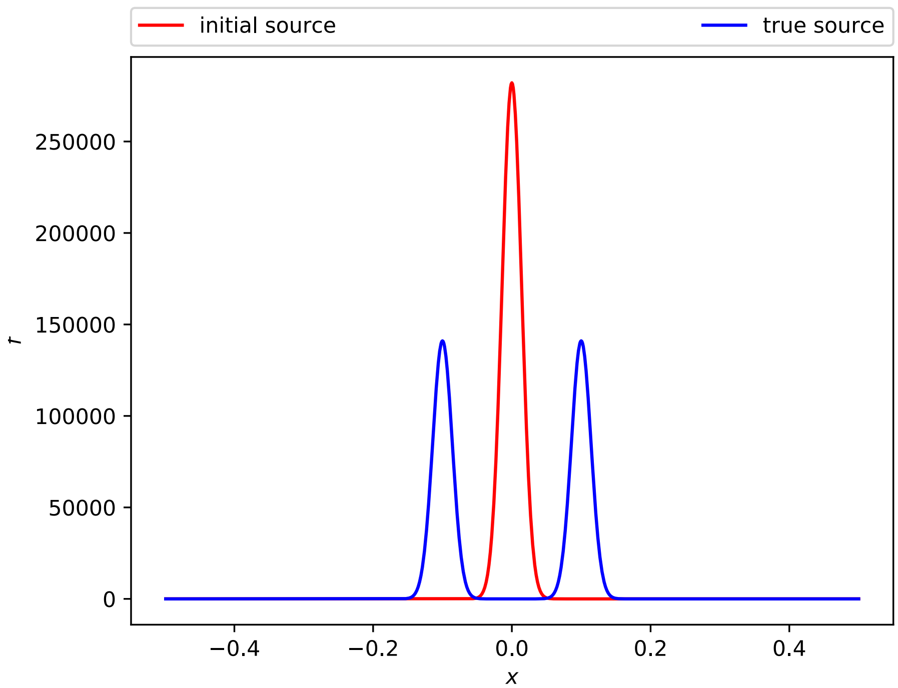

FIGURE 1. Blue: true source $f(x)$ that needs to be recovered using the numerical method. Red: Initial guess for the source (first iterate).

We note that we were able to recover f and ψ uniquely without regularizing, which is not the case for many inverse problems. f has only two derivatives less than u in space and one derivative less in time. We measure u at a certain time T, and to recover f, we are not adding an infinitely many derivatives as in the case of the problem of recovering initial conditions. In those problems, we would have lost a lot of detail by the time we observe u. In the source recovery problem, if f is not smooth, we'll observe that immediately in the second derivative of the function, since the source is embedded in the solution, not lost like the initial conditions. The source function f appears in the analytic solution, integrated against the kernel, but with zero time lag. The same applies for the velocity ψ. This makes the case for no-regularization-required in order to recover the source.

The following questions are to be explored next:

(1) The requirement that f has compact support is crucial for our method. Can we have a similar result by relaxing the assumption that f has compact support? A first step would be to consider source functions that can be suitably approximated by compactly supported functions.

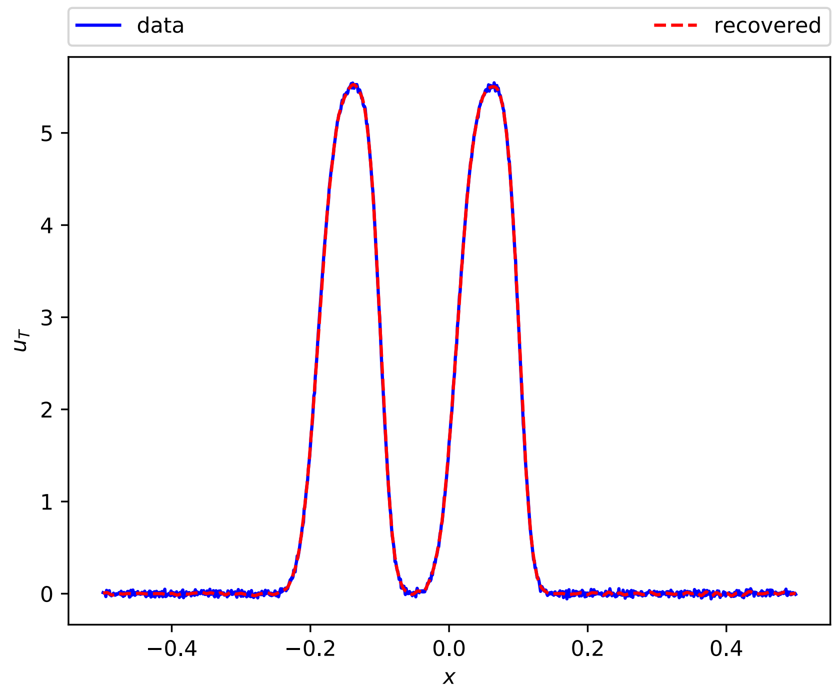

FIGURE 2. Blue: Data obtained by solving the advection diffusion equation with source function given in Figure 1. Red: Recovered data using the numerical method.

(2) Is this the most we can do by exploiting one measurement? Are there non-constant $\psi_1 \neq \psi_2$ and $f_1 \neq f_2$ compactly supported such that $u_1(\cdot, T) = u_2(\cdot, T)$? Can Carleman estimates techniques be used to answer this question?
(3) Can we get uniqueness using two measurements $u(\cdot, T_1)$ and $u(\cdot, T_2)$, for more general types of source functions?

6. Acknowledgements

This problem was suggested to us by Selim Esedoglu at the University of Michigan. Hala AH Shehadeh was partially supported by the James Madison University Program of Grants for Faculty Assistance.

Appendix A.

In this appendix we state the theorems that are necessary to establish the results of this paper.

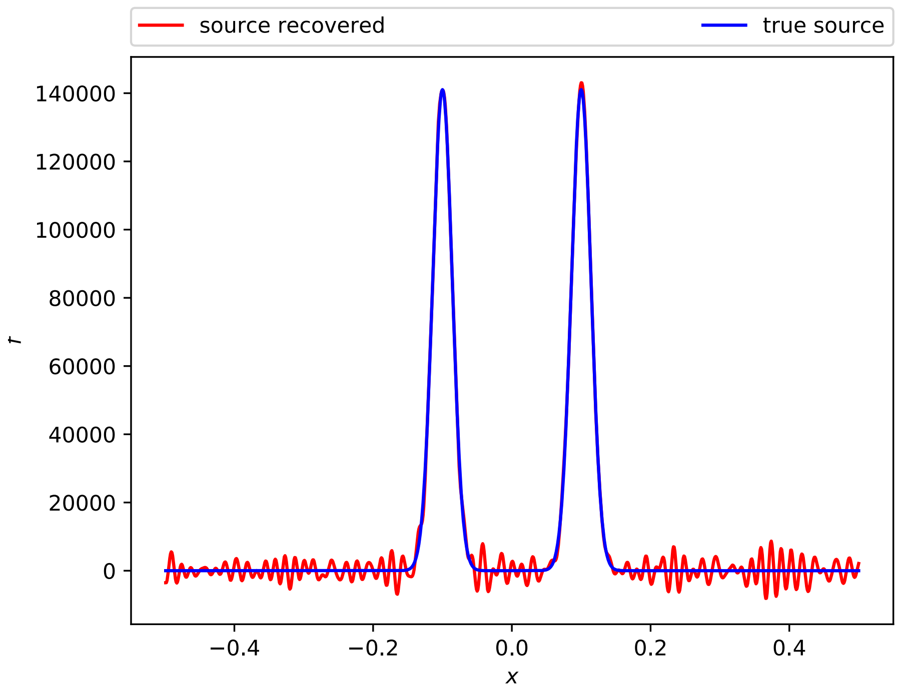

FIGURE 3. Recovered source function vs true source function.

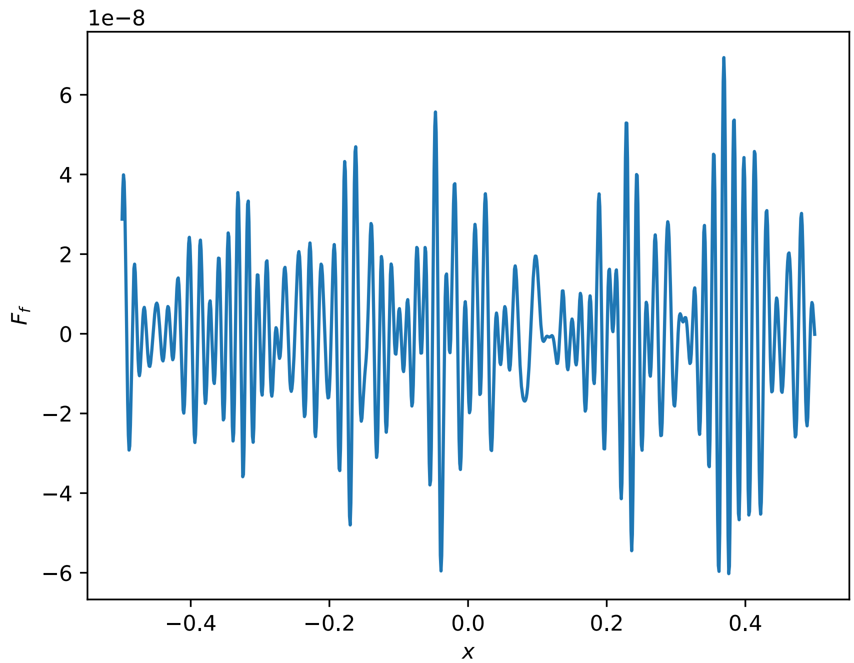

FIGURE 4. The derivative of the functional F with respect to f is zero meaning that the numerical simulation has converged to a minimizer.

Theorem 4 (Paley-Wiener)**.** *Let $f(z)$ be of the form*

$$f(z) = \int_{-A}^{A} F(t)e^{itz}dt \tag{42}$$

where $0 < A < \infty$ and $F \in L^2(-A, A)$. Then f is entire and it satisfies the growth condition,

$$|f(z)| \leq Ce^{A|z|} \tag{43}$$

with $C < \infty$. Moreover, the restriction of f to the real axis lies in $L^2(\mathbb{R})$.

Conversely, if A and C are positive constants and f is an entire function such that

$$|f(z)| \leq Ce^{A|z|} \tag{44}$$

for all $z \in \mathbb{C}$, and if $f \in L^2(\mathbb{R})$, then there exists an $F \in L^2(-A, A)$ such that

$$f(z) = \int_{-A}^{A} F(t)e^{itz}dt \tag{45}$$

for all $z \in \mathbb{C}$.

The proof of the above theorem can be found in standard real and complex analysis or Fourier analysis books (see [1] for example).

The next theorem is known as the projection slice theorem, the central section theorem, or the Fourier slice theorem. It is actually a very important theorem in computed tomography. In d-dimensions, it states that the Fourier transform of the projection of a d-dimensional function onto an m-dimensional linear submanifold is equal to an m-dimensional slice of the Fourier transform of that function. The following variation, which is used in this paper, says that the Fourier transform of the integral of a function over a hyperplane is equal to the slicing of the Fourier transform of that function on the hyperplane.

Theorem 5 (Projection Slice)**.** *Let $s \in \mathbb{R}$, and ν a unit vector normal to a hyperplane $x \cdot \nu = s$ in $\mathbb{R}^d$. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be such that $f \in L^1(\mathbb{R}^d)$. Define the projection operator*

$$P_\nu[f](s) = \int_{x \cdot \nu = s} f(x)dA, \tag{46}$$

which integrates f over the hyperplane perpendicular to ν. Define the slice operator

$$S_\nu[f](s) = f(s\nu), \tag{47}$$

which evaluates f at the point $s\nu$. Then we have

$$\widehat{P_\nu[f]}(\xi) = S_\nu[\hat{f}](\xi) = \hat{f}(\xi\nu), \tag{48}$$

where $\xi \in \mathbb{R}$.

Proof.

$$
\begin{aligned}
\widehat{P_\nu[f]}(\xi) &= \int_{-\infty}^{\infty} e^{is\xi} \int_{x\cdot\nu=s} f(x)dAds \\
&= \int_{-\infty}^{\infty} \int_{x\cdot\nu=s} e^{i(x\cdot\nu)\xi} f(x)dAds \\
&= \int_{\mathbb{R}^n} e^{i(x\cdot\nu)\xi} f(x)dx \\
&= \int_{\mathbb{R}^n} e^{ix\cdot(\xi\nu)} f(x)dx \\
&= \hat{f}(\xi\nu).
\end{aligned} \tag{49}
$$

The third equality is due to the fact that $\mathbb{R}^n$ is sliced into all parallel hyperplanes $x \cdot \nu = s$, where $s \in (-\infty, \infty)$. □

References

[1] W. Rudin, Real and Complex Analysis (Third Edition), McGraw-Hill Education, 1986.

[2] F. Tröltzsch, Optimal Control of Partial Differential Equations, AMS Graduate Studies in Mathematics, 2010.

[3] J. R. Cannon and P. DuChateau, *Structural identification of an unknown source term in a heat equation* Inverse Probl. **14** (1998), 535.

[4] X. Li and S. Qian, *Numerical Solution of the Inverse Problem of Determining an Unknown Source Term in a Heat Equation*, J. Appl. Math. **2012** (2012), 390876.

[5] A. Hamdi, *Identification of point sources in two dimensional advection-diffusion-reaction equation: Application to pollution sources in a river. Stationary case*, Inverse Probl. Sci. Eng. **15 (8)** (2007): 855-870..

[6] A. Hasanov and B. Pektas, *Identification of an unknown time-dependent heat source term from overspecified Dirichlet boundary data by conjugate gradient method*, J. Comput. Math. Appl. **65** (2013), 42-57.

[7] V. Isakov, Inverse Problems for Partial Differential Equations (Second Edition), Springer, 2005.

[8] A. Dutt and V. Rokhlin, *Fast Fourier transforms for non-equispaced data*, SIAM J. Sci. Comput. **14** 6 (1993), 1368-1393.

[9] NUFFT page link http://www.cims.nyu.edu/cmcl/nufft/nufft.html

[10] Foreman-Mackey interface link https://github.com/dfm/python-nufft

[11] Link for code https://github.com/vidalalcala/source-recovery/blob/master/advection_inverse.py

[12] Link for code https://github.com/vidalalcala/source-recovery/blob/master/advection_solver.py