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METHOD FOR SOLVING NONLINEAR INITIAL VALUE PROBLEMS BY COMBINING HOMOTOPY PERTURBATION AND FUZZY REPRODUCING KERNEL HILBERT SPACE METHODS

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Abstract. In the previous works, the authors presented the fuzzy reproducing kernel method(FRKM) for solving various boundary value problems. The nonlinear singular initial value problems including generalized Lane − Emden−type equations are investigated by combining homotopy perturbation method (HPM) and fuzzy reproducing kernel Hilbert space method (FRKHSM). He's HPM is based on the use of traditional perturbation method and homotopy technique and can reduce a nonlinear problem to some linear problems and generate a rapid convergent series solution in most cases.FRKHSM is also an analytical technique, which can overcome the difficulty at the singular point of non-homogeneous, linear singular initial value problems; especially when the singularity appears on the right-hand side of this type of equations, so it can solve powerfully linear singular initial value problems. Therefore, using advantages of these two methods, more general nonlinear singular initial value problems can be solved powerfully.Some numerical examples are presented to illustrate the strength of the method.

Key words and phrases. Fuzzy Reproducing kernel method, Linear boundary value problems,Homotopy perturbation method.

1. Introduction

Singular initial value problems in ordinary differential equations occur in several models of non-Newtonian fluid mechanics, mathematical physics, astrophysics, etc. [1,2]. For example, the theory of internal structure of stars, cluster of galaxies, thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and theories of thermionic currents are modeled by means of Lane−Emden equations. The main idea of this paper is to present an algorithm for computing the solutions of singular initial value problems including

Lane−Emden-type equations of the form

$$
(1.1) \qquad \begin{cases} u''(x) + \frac{k_1}{a(x)}u'(x) + \frac{k_2}{b(x)}u(x) + f(x,u) = g(x), \quad 0 < x \leq 1, \\ u(0) = \alpha, u'(0) = \beta. \end{cases}
$$

where α, β, k_1, k_2 are real constants, $a(x), b(x)$, are continuous and maybe $a(0) = 0, b(0) = 0, f(x,y)$ is a continuous real valued function, and $g(x) \in c(0,1]$, i. e., the case when the function $g(x)$ may be undefined at the origin. Such problems have attracted much attention and have been studied by many authors [3−12]. There is considerable interest in numerical methods on singular initial value problems. Recently, Ramose [4,5] has developed linearization methods for the numerical solution of (1.1) with $k_1 = 2, a(x) = xandk_2 = 0$. By applying the Adomian decomposition method, Wazwaz [6−8] has also investigated the special case of singular initial value problem (1.1). have solved (1.1) with $k_1 = 2, a(x) = x, k_2 = 0andg(x) \in c[0,1]$ by using He's homotopy perturbation method and variational iteration method. In these references, the authors selected $g(x)$ as a well-defined function at the origin. In [11], This switching from operations of calculus to algebraic operations on transforms is called operational calculus, a very important area of applied mathematics, and for the engineer, the fuzzy reproducing kernel Hilbert space methodsis practically the most important operational method. The fuzzy reproducing kernel Hilbert space methods also has the advantage that it solves problems directly without determining a general solution in the first and obtaining non homogeneous differential equations in the second. One can see some useful papers about fuzzy reproducing kernel Hilbert space methods in [16, 27]. Also, there exist some recently published papers with some modifications about application of fuzzy reproducing kernel Hilbert space methods to solve fuzzy differential equation [30, 31]. In recent years, the homotopy perturbation method (HPM), first proposed by He [23], has successfully been applied to solve many types of linear and non- linear functional equations. This method which is a combination of homotopy in topology, and classic perturbation techniques, provides a convenient way to obtain analytic or approximate solutions to a wide variety of problems arising in different fields, see [15, 17, 21, 23, 24, 25] and the references there.

In this work, we intend to use HPM for computing the fuzzy reproducing kernel Hilbert space methods.In this work, we will give the analytic approximation of the general singular initial value problem (1.1) by combining He's HPM and FRKHSM. The HPM was proposed originally by He. This method is based on the use of traditional perturbation method and homotopy technique. Using this method, a rapid convergent series solution can be obtained in most cases. Usually, a few number of terms of the series solution can be used for numerical purposes with a high degree of accuracy. The method was successfully applied to

boundary value problems, partial differential equations and other fields .Fuzzy reproducing kernel theory has important application in numerical analysis, Fuzzy differential equation, probability and statistics and so on . In this work, we intend to use HPM for computing the fuzzy reproducing kernel Hilbert space methods. The paper is organized as follows:
In section 2, Reproducing kernel method. In section 3,we present the basic notions of fuzzy number, fuzzy valued function, fuzzy derivative, fuzzy integral and fuzzy reproducing kernel Hilbert space methods. In section 4,we present the basic notionshomotopy perturbation method . In section 5. homotopy perturbation method is presented and the procedure for computing the fuzzy reproducing kernel Hilbert space methods is proposed by proving some theorems based on the HPM and finally som examples are given. Conclusions are drawn in section 6.

2. Reproducing kernel method

In this section, we illustrate how to solve the following linear singular initial value problem using RKHSM:

$$\begin{cases} u''(x) + \frac{k_1}{a(x)}u'(x) + \frac{k_2}{b(x)}u(x) + f(x,u) = g_0(x), & 0 < x \leq 1, \\ u(0) = 0, u'(0) = 0. \end{cases} \tag{2.1}$$

where α, β, k_1, k_2 are real constants, $u(x) \in W_2^3[0,1]$, $a(x), b(x)$ are continuous and mayby $a(0) = 0$, $b(0) = 0$,and $g_0(x) \in c(0,1]$, i. e., the case when the function $f_0(x)$ may be undefined at the origin. Multiplying both sides of (3.1) by $a(x), b(x)$, we have

$$\begin{cases} Lu(x) = f(x), & 0 < x \leq 1, \\ u(0) = \alpha, u'(0) = \beta, \end{cases} \tag{2.2}$$

In (5.3), put $Lu(x) = a(x)b(x)u^{''}(x) + k_1 b(x)u^{'}(x) + k_2 a(x)u(x), f(x) = a(x)b(x)g_0(x)$.
It is clear that $L : W^m[0,1] \to W^1[0,1]$ is a bounded linear operator.because $W^1[0,1]$ is a reprodusing kernel space, there exists a reprodusing kernel $\bar{k}(x,y)$. for every linear operator L,
$\mid f(x) \mid = \mid Lu(x) \mid = \mid (Lu(y), k(x,y))_{W^m} \mid \leq \| f(y) \|_m \| k(x,y) \|_m \leq c_0 \| f \|_m$, Put $\varphi_i(x) = \bar{k}(x_i, y)$ and $\psi_i(x) = L^*\varphi_i(x)$ where $\bar{k}(x_i, y)$ is the reproducing kernel of $W^1[0,1], L^*$ is the adjoint operator of L. The orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ of $W^m[0,1]$ can be derived from the Gram-Schmidt orthogonalization process of $\psi_i(x)_{i=1}^{\infty}$,

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \qquad (\beta_{ii} > 0,\ i = 1, 2, \ldots). \tag{2.3}$$

where $\beta_{ik}\{(i = 1, 2, \ldots), (k = 1, 2, \ldots)\}$ are coefficients of Gram-Schmidt orthonormalizarion and $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ is an orthonormal system,could be determined by solving the following equations.

$B_{ii} = \sum_{i=0}^{m-1} \psi_i^{(i)}(0)\bar{\psi}_i^{(i)}(0) + \int_0^1 \psi_i^{(m)}(x)\bar{\psi}_i^{(m)}(x)\mathrm{d}x$

$\beta_{ii} = 1/(\sqrt{[\sum_{i=0}^{m-1}(\psi_i^{(i)}(0))^2 + \int_0^1(\psi_i^{(m)}(x))^2\mathrm{d}x - \sum_{i=1}^{i-1} B_{ii}]})$

$\beta_{ij} = \beta_{ii} * (-\sum_{k=j}^{i-1} B_{ik} * \beta_{kj}) \qquad (i = 1, 2, \ldots), \qquad (j = 1, 2, \ldots, i-1), \qquad (k = 1, 2, \ldots, i-1)$. In order to solve(3.2) using RKHSM, we first construct a reproducing kernel Hilbert space $W_2^3[0,1]$ in which every function satisfies the initial conditions of (3.1).

Definition 2.1. The inner product space $W_2^3[0,1]$ is defined as $W_2^3[0,1] = \{u(x)|u, u', u'' are\ absolutely\ continuous\ real\ value\ functions, u^{(3)} \in L^2[0,1], u(0) = 0, u'(0) = 0\}$. The inner product in $W_2^3[0,1]$ is given by
$\langle u(y), v(y)\rangle = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + \int_0^1 u^{(3)}v^{(3)}dy$,
and the norm$\|u\|_W = \sqrt{\langle u(y), u(y)\rangle}$,where $u, v \in W_2^3[0,1]$

According to [1, 6], we have the following theorem:

Theorem 2.0.1. *For* (5.1)*, if $\{x_i\}_i^{\infty}$ is dense on [0,1], then $\psi_i(x)_{i=1}^{\infty}$ is the complete sestem of $W^m[0,1]$ and $\psi_i(x) = L_s k(x,s)|_{s=x_i}$.*

Proof.
From $\varphi_i(x) = \bar{k}(x_i, y)$ and $\psi_i(x) = L^*\varphi_i(x)$ we have $\psi_i(x) \in W^m[0,1] \qquad (i = 1, 2, \ldots)$.
on the other hand, for any $u(x) \in W^m[0,1]$, it has
, soppose, $(u(x), \psi_i(x))$=0
namely $(u(x), (L^*\phi_i(x)))$=$(Lu(.), \phi_i(.))$=$(Lu)(x_i))$=0
Since $\{x_i\}_i^{\infty}$ is dense on [0,1] so that (Lu)(x)=0
yield u(x)=0
and $\psi_i(x) = (L^*\phi_i)(x)$=$((L^*\phi_i)(s), k(x,s))$=$(\phi_i(s), L_s k(x,s))$=$L_s k(x,s)$
The proof is complete.

Theorem 2.0.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense on[0,1] and the solution of*
Proof.

$$\begin{cases} u''(x) + p(x)u'(x) + q(x)u(x) = f(x), & \textit{if } 0 < x < 1; \textit{ (1.1)} \\ u(0) = 0, u(1) = 0 \end{cases}$$

is unique, then the solution of (5.1) *is*

$$u(x) = \sum_{j=1}^{\infty} A_j \overline{\psi}_j(x) \tag{2.4}$$

where

$$A_j = \sum_{l=1}^{j} \beta_{jl} f(x_l)$$

Proof.

$$u(x) = \sum_{j=1}^{\infty} \langle u(x), \overline{\psi}_j \rangle \overline{\psi}_j(x) = \sum_{j=1}^{\infty} \sum_{l=1}^{j} \beta_{jl} \langle u(x), L^* \varphi_j(x) \rangle \overline{\psi}_j(x)$$

$$= \sum_{j=1}^{\infty} \sum_{l=1}^{j} \beta_{jl} \langle Lu(x), \varphi_j(x) \rangle \overline{\psi}_j(x)$$

$$= \sum_{j=1}^{\infty} \sum_{l=1}^{j} \beta_{jl} \langle f(x), \varphi_j(x) \rangle \overline{\psi}_j(x)$$

$$= \sum_{j=1}^{\infty} \sum_{l=1}^{j} \beta_{jl} f(x_l) \overline{\psi}_j(x) = \sum_{j=1}^{\infty} A_j \overline{\psi}_j(x)$$

□

Now, the approximate solution u(x) can be obtained by taking finitely many terms in the series representation of u(x) and

$$u_N(x) = \sum_{j=1}^{N} A_j \overline{\psi}_j(x) \tag{2.5}$$

Remark 2.0.3. *Since $W^m[0,1]$ is a Hilbert space,ifu is solution of Lu=f*

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k) \overline{\psi}_i(x)$$

Since $\{\overline{\psi}_i\}_{i=1}^{\infty}$ is a normal basis for $W^m[0,1]$. the square sum of the Fourier coefficients of u *is convergent.*

$$\sum_{i=1}^{\infty}(\sum_{k=1}^{i}\beta_{ik}f(x_k))^2 < \infty$$

Therefore, the sequence u_N is convergent in the sense of norm $\| \,.\, \|$

$$\| u_N(x) \| = \| \sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik}f(x_k)\overline{\psi}_i(x) \| \leq \| \sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik}f(x_k) \|$$

Lemma 2.0.4. *If $u(x) \in W^m[0,1]$, then there exists a constant c such that $| u(x) | \leq c \| u(x) \|_m$, $| u^{(k)}(x) | \leq c \| u(x) \|_m, 1 \leq k \leq m-1$.*

Proof. Since
$| u(x) | = | (u(y), k(x,y))_m | \leq \| u(y) \|_m \| k(x,y) \|_m$,
there exists a constant c_0 such that
$c_0 = \| k(x,y) \|_m \in W^m[0,1]$
$| u(x) | \leq c_0 \| u \|_m$
Note that
$| u^{(i)}(x) | = | (u(y), \frac{\partial^i k(x,y)}{\partial x^i})_m |$
$\leq \| u \|_4 \| \frac{\partial^i k(x,y)}{\partial x^i} \|_m$
$\leq c_i \| u \|_m, (i = 0, 1, 2, ..., m-1)$,
where c_i are constants.
Putting $c = max_{0 \leq i \leq m-1} c_i$ and the proof of the lemma is complete.

Theorem 2.0.5. *The approximate solution $u_n(x)$ and its derivatives $u_n^{(k)}(x), 1 \leq k \leq m-1$ are all uniformly convergent.*

Proof. We know
$u_n(x) - u(x) = (u_n(y) - u(y), k(x,y))_{W^m}$
$u_n^k(x) - u^k(x) = (u_n(x) - u(x))^{(k)} = \frac{\partial^k}{\partial x^k}((u_n(y) - u(y), k(x,y))_{W^m})$
$= (u_n(y) - u(y), \frac{\partial^k}{\partial x^k} k(x,y))_{W_m}$
$\frac{\partial^k}{\partial x^k} k(x,y) \in W^m[0,1]$, one obtains
$| u_n^k(x) - u^k(x) | \leq \| u_n(y) - u(y) \|_{W^m} \| \frac{\partial^k}{\partial x^k} k(x,y) \|$
Also
$\| \frac{\partial^k}{\partial x^k} k(x,y) \|_{W_m}$ is continuous with respect to x in $[0,1]$, then

$| u_n^k(x) - u^k(x) |\leq M \parallel u_n(y) - u(y) \parallel_{W_m}$
where M is a positive number.
So that
$\lim_{x\to n} u_n(x) = u(x) \Rightarrow \lim_{x\to n} u_n^k(x) = u^k(x)$. **because**$1 \leq k \leq m-1 \in W^m[0,1]$

Theorem 2.0.6. *The space $W_2^3[0,1]$ is a reproducing kernel Hilbert space. That is, there exists $R_x(y) \in W_2^3[0,1]$, for any $u(y) \in W_2^3[0,1]$ and each fixed $x \in [0,1], y \in [0,1]$, such that $(u(y), R_x(y))_{W_2^3} = u(x)$. The reproducing kernel $R_x(y))$ can be denoted by*

$$\begin{cases} \frac{y^2(-(x^2(-126+10x-5x^2+x^3))+5(-1+x)xy^2-(-1+x^2)y^3)}{120}, & y \leq x, \\ \frac{-(x^2(-5x^2(-1+y)y+x^3(-1+y^2)+y^2(-126+10y-5y^2+y^3)))}{120}, & y > x. \end{cases} \tag{2.6}$$

For the proof of this theorem and the method of obtaining reproducing kernel $R_x(y)$,refer to [37, 38].

3. Fuzzy set

We represent an arbitrary fuzzy number by an ordered pair function $(\underline{u}(r), \overline{u}(r))$, which satisfies the following requirements [26]:
a: $\underline{u}(r)$ is abounded monotonic increasing left continuous function,
b: $\overline{u}(r)$ is abounded monotonic decreasing left continuous function,
c: $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.
A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$. We recall that for $a < b < c$ which $a, b, c \in R$, the triangular fuzzy number $u = (a, b, c)$ are determined by a, b, c such that $\underline{u}(r) = a + (b-c)r$ and $\overline{u}(r) = c - (c-b)r$ are the endpoints of the r-level sets, for all $r \in [0,1]$. Let E be the set of all fuzzy number on $\Re$.
For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $k > 0$, we define addition $u \oplus v$, subtraction $u \ominus v$ and scalar multiplication by k as [22].
a) Addition:
$u \oplus v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,
b) subtraction:
$u \ominus v = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r))$,
c) scalar multiplication:

$$k \odot u = \begin{cases} (k\underline{u}, k\overline{u}) & k \geq 0 \\ & , \\ (k\overline{u}, k\underline{u}), & k < 0 \end{cases}$$

if $k = -1$ then $k \odot u = -u$.

Definition 3.1. [29] For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$, we show the Hausdorff distance between u and v by $D(u, v)$, and take $D : E \times E \longrightarrow \Re_+ \cup (0)$. Also, we know (E, D) is a complete metric space, thus:

$$D(u, v) = \sup_{0 \le r \le 1} \{\max[|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|]\},$$

We have following traits for Hausdorff distance; per $u, v, e, f \in E$ and all $k \in \Re$:
i) $D(u + e, v + e) = D(u, v)$,
ii) $D(ku, kv) = |k|D(u, v)$,
iii) $D(u + v, e + f) \le D(u, e) + D(v, f)$.

Definition 3.2. [22] Let $f : \Re \longrightarrow E$ be a fuzzy-valued function. If for arbitrary fixed $x_0 \in \Re$ and $\epsilon > 0$, a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow D(f(x), f(x_0)) < \epsilon,$$

f is said to be continuous.

Definition 3.3. [32] A mapping $f : \Re \times E \longrightarrow E$ is called continuous at point $(t_0, x_0) \in \Re \times E$ provided for any fixed $r \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists an $\delta(\epsilon, r) > 0$, such that

$$D([f(t, x)]_r, [f(t_0, x_0)]_r) < \epsilon,$$

whenever $|t - t_0| < \delta$ and $D([x]_r, [x_0]_r) < \delta(\epsilon, r)$ for all $t \in \Re,\ x \in E$.

Theorem 3.0.1. [33] *Let $f(x)$ be a fuzzy value function on $[a, b]$ and it is represented by $(\underline{f}(x, r), \overline{f}(x, r))$ for $r \in [0, 1]$, assume $\overline{f}(x, r)$ are Riemann-integrable on $[a, b]$ for every $b \ge a$ and assume. There are two positive values $\underline{M}(r)$ and $\overline{M}(r)$ such that*

$$\int_a^b |\underline{f}(x, r)|dx \le \underline{M}(r) \ ,$$

and

$$\int_a^b |\overline{f}(x, r)|dx \le \overline{M}(r) \ ,$$

for every $b \ge a$.
Then $f(x)$ is improper fuzzy Riemann-integrable on $[a, \infty)$ and is a fuzzy number. furthermore, we have:

$$\int_a^\infty f(x)dx = (\int_a^\infty \underline{f}(x)dx, \int_a^\infty \overline{f}(x)dx) \ .$$

Proposition 1. [34] If $f(x)$ and $g(x)$ are fuzzy value functions and fuzzy Riemann-integrable on $[a,\infty)$ then $f(x)+g(x)$ is fuzzy Riemann-integrable on $[a,\infty)$.
Moreover , we have:

$$\int_l (f(x)\oplus g(x))dx = \int_l f(x)dx \oplus \int_l g(x)dx \ .$$

It is well-known that the H-derivative (differentiability in the sense of Hukuhara) for fuzzy mappings was initially introduced by Puri and Ralescu [28] and it is based in the H-difference of sets, as follows.

Definition 3.4. Suppose $x, y \in E$. If there exists $z \in E$ such that $x = y \oplus z$, then z is called the H-difference of x and y, and it is denoted by $x -^h y$.
In this paper, the sign "$-^h$" always stands for H-difference and also note that $x -^h y \neq x \ominus y$.
We consider the following definition which was introduced by Bede et al. [18].

Theorem 3.0.2. [20] *Let $f : R \longrightarrow E$ be a function and denote $f(t) = (\underline{f}(t,r), \overline{f}(t,r))$ for each $r \in [0,1]$. Then*
(1) If f is (i)-differentiable, then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and $f'(t) = (\underline{f}'(t,r), \overline{f}'(t,r))$,
(2) If f is (ii)-differentiable, then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and $f'(t) = (\overline{f}'(t,r), \underline{f}'(t,r))$.

Definition 3.5. .Let Ω be the universal set, A fuzzy set on Ω , is the set $X \subset \Omega$ with membership function
$\mu\mathbf{x} : \Omega \to [0,1], \mathbf{x} \to \mu\mathbf{x}(x)$

Definition 3.6. .$(\alpha - cut\ of\ a\ fuzzy\ set)$.
Theα-cut of a fuzzy set $X \subset \Omega$ is the set $X_\alpha = \{x \in \Omega | \mu\mathbf{x}(x) \geqslant \alpha, \alpha \in [0,1]\}$.

Definition 3.7. . T-Norm
A triangular norm or T-norm is the function $T : [0,1]^2 \to [0;1]$, that for all $x;y;z \in [0;1]$satisfy:
T_1commutativity: $T(x;y) = T(y;x)$;
T_2associativity: $T(x;T(y;z)) = T(T(x;y);z)$;
T_3 monotonicity:$(y \leqslant z) \Longrightarrow T(x;y) \leqslant T(x;z)$;
T_4 boundary condition $T(x;1) = x$.

Using $n \in N$and associativity, a multiple-valued extension
$T_n : [0; 1]^n \to [0; 1]$ of a T-norm T is given by $T_n(x_1; x_2; \ldots; x_n) = T(x_1; T_{n-1}(x_2; x_3; \ldots; x_n)$:

We will use $T$to denote Tor T_n Intersection kernel on Fuzzy Sets The intersection of two fuzzy sets
$X; Y \in F(s \subset \Omega)$ is the fuzzy set$X \cap Y \in F(s \subset \Omega)$with membership function
$\mu X \cap Y : \Omega \to [0, 1]$
$x \to \mu X \cap Y = T(\mu X(x), \mu Y(x))$
where T is a T-norm operator. Using this fact, we define the intersection kernel on fuzzy sets as follows:

Definition 3.8. .(Intersection Kernel on Fuzzy Sets). Let $X; Y$ be two fuzzy sets in $F(s \subset \Omega)$, the intersection kernel on fuzzy sets is the function
$k : F(s \subset \Omega) \times F(s \subset \Omega \to R,\ (X, Y) \to k(X, Y) = g(X \cap Y)$
where g is the mapping $g : F(s \subset \Omega) \to [0, \infty],\ X \to g(X)$.
The mapping g plays an important role assigning real values to the intersection fuzzy set $X \cap Y$. We can think about this function as a similarity measure between two fuzzy sets and its design will be highly dependent of the problem and the data. For instance, our first choice for g uses the fact that the support of $X \cap Y$, has finite decomposition, that is, $(X \cap Y)_{>0} = \cup_{i \in I} A_i \in s$
of pairwise disjoint sets $\{A_1; A_2; \ldots; A_N\}$. We can measure its support using the measure$\rho : s \to [0, \infty]$ as follows:
$\rho((X \cap Y)_{>0}) = \rho(\cup_{i \in I}) = \sum_{i \in I} \rho(A_i)$ The idea to include fuzziness is to weight each$\rho(A_i)$by a value given by the contribution of the membership function on all the elements of the setA_i. Next, we give a definition of a intersection kernel on fuzzy sets using the concept of measure and membership function.

Definition 3.9. (Intersection Kernel on Fuzzy Sets with measure ρ)
Let $\cup_{i \in I} A_i \in s$ a finite decomposition of the support of the intersection fuzzy set$(X \cap Y) \in F(s \subset \Omega)$ as defined before. Let g be the function $g : F(s \subset \Omega) \to [0, \infty]$

$$(X \cap Y) \to g(X \cap Y) = \sum_{i \in I} \mu X \cap Y(A_i) \rho(A_i)$$

where

$$\mu X \cap Y(A_i) = \sum_{x \in A_i} \mu X \cap Y(x)$$

We define the Intersection Kernel on Fuzzy Sets with measure ρ as

$$k(X,Y) = g(X \cap Y) = \sum_{i \in I} \mu X \cap Y(A_i)\rho(A_i)$$

Using the T-norm operator, the intersection kernel on fuzzy sets with measure ρ given by (12) can be written as

$$\begin{aligned} k(X,Y) = \sum_{i \in I} \mu X \cap Y(A_i)\rho(A_i) &= \sum_{i \in I} \sum_{x \in A_i} \mu X \cap Y(x)\rho(A_i) \\ &= \sum_{i \in I} \sum_{x \in A_i} T(\mu X(x), \mu Y(x))\rho(A_i) \end{aligned}$$

The next step is to determine which intersection kernels on fuzzy sets with measure ρ are positive definite, that is, which intersection kernels are reproducing kernels of some RKHS.

Definition 3.10. $\underline{W}_2^3[0,1] = \{\underline{u}(x,r)|\underline{u}(t,r), \underline{u}'(t,r), \underline{u}''(t,r), absolutely\ continuous\ real\ value,$ $\underline{u}^{(3)}(t,r) \in L^2[0,1], \underline{u}(0,r) = 0, \underline{u}'(0,r) = 0\}$.
Also
$\overline{W}_2^3[0,1] = \{\overline{u}(x,r)|\overline{u}(t,r), \overline{u'}(t,r), \overline{u''}(t,r), absolutely\ continuous\ real\ value, \overline{u}^{(3)}(t,r) \in L^2[0,1],$ $\overline{u}(0,r) = 0, \overline{u'}(0,r) = 0\}$.
The inner product in $(\underline{W}_2^3[0,1], \overline{W}_2^3[0,1])$ is given by

$$\langle \underline{u}(y,r), \underline{v}(y,r) \rangle = \underline{u}(0,r)\underline{v}(0,r) + \underline{u}'(0,r)\underline{v}'(0,r) + \underline{u}(1,r)\underline{v}(1,r) + \int_0^1 \underline{u}^{(3)}(y,r)\underline{v}^{(3)}(y,r)dy,$$

and the norm

$$\|\underline{u}(y,r)\|_{\underline{W}} = \sqrt{\langle \underline{u}(y), \underline{u}(y) \rangle}, where \underline{u}, \underline{v} \in \underline{W}_2^3[0,1]$$

and

$$\langle \overline{u}(y,r), \overline{v}(y,r) \rangle = \overline{u}(0,r)\overline{v}(0,r) + \overline{u'}(0,r)\overline{v'}(0,r) + \overline{u}(1,r)\overline{v}(1,r) + \int_0^1 \overline{u}^{(3)}(y,r)\overline{v}^{(3)}(y,r)dy,$$

and the norm

$$\|\overline{u}(y,r)\|_{\overline{W}} = \sqrt{\langle \overline{u}(y), \overline{u}(y) \rangle}, where \overline{u}, \overline{v} \in \overline{W}_2^3[0,1]$$

therefore

$$(\langle \underline{u}(y,r), \underline{v}(y,r)\rangle, \langle \overline{u}(y,r), \overline{v}(y,r)\rangle) =$$

$$(\underline{u}(0,r)\underline{v}(0,r) + \underline{u}'(0,r)\underline{v}'(0,r) + \underline{u}(1,r)\underline{v}(1,r) + \int_0^1 \underline{u}^{(3)}(y,r)\underline{v}^{(3)}(y,r)dy,$$

$$\overline{u}(0,r)\overline{v}(0,r) + \overline{u'}(0,r)\overline{v'}(0,r) + \overline{u}(1,r)\overline{v}(1,r) + \int_0^1 \overline{u}^{(3)}(y,r)\overline{v}^{(3)}(y,r)dy)$$

and

$$(\|\underline{u}(y,r)\|_{\underline{W}}, \|\overline{u}(y,r)\|_{\overline{W}}) = (\sqrt{\langle \underline{u}(y), \underline{u}(y)\rangle}, \sqrt{\langle \overline{u}(y), \overline{u}(y)\rangle})$$

Definition 3.11. The orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ of $W^m[0,1]$ can be derived from the Gram-Schmidt orthogonalization process of $\{\underline{\psi}_i(x,r), \overline{\psi}_i(x,r)\}_{i=1}^{\infty}$,

$$(3.1)\quad (\underline{\bar{\psi}}_i(x,r), \overline{\bar{\psi}}_i(x,r)) = (\sum_{k=1}^{i} \underline{\beta}_{ik}\underline{\psi}_k(x,r), \sum_{k=1}^{i} \overline{\beta}_{ik}\overline{\psi}_k(x,r)), \qquad (\underline{\beta}_{ii}, \overline{\beta}_{ii}) > 0,\ i = 1,2,\ldots.$$

where $(\underline{\beta}_{ik}, \overline{\beta}_{ik})\{(i = 1,2,\ldots), (k = 1,2,\ldots)\}$ are coefficients of Gram-Schmidt orthonormalizarion and $\{(\underline{\bar{\psi}}_i(x,r), \overline{\bar{\psi}}_i(x,r))\}_{i=1}^{\infty}$ is an orthonormal system,could be determined by solving the following equations.

$$\underline{B}_{ii} = \sum_{i=0}^{m-1} \underline{\psi}_i^{(i)}(0,r)\underline{\bar{\psi}}_i^{(i)}(0,r) + \int_0^1 \underline{\psi}_i^{(m)}(x,r)\underline{\bar{\psi}}_i^{(m)}(x,r)\mathrm{d}x$$

and

$$\overline{B}_{ii} = \sum_{i=0}^{m-1} \overline{\psi}_i^{(i)}(0,r)\overline{\bar{\psi}}_i^{(i)}(0,r) + \int_0^1 \overline{\psi}_i^{(m)}(x,r)\overline{\bar{\psi}}_i^{(m)}(x,r)\mathrm{d}x$$

$$\underline{\beta}_{ii} = 1/(\sqrt{[\sum_{i=0}^{m-1}(\underline{\psi}_i^{(i)}(0,r))^2 + \int_0^1 (\underline{\psi}_i^{(m)}(x,r))^2\mathrm{d}x - \sum_{i=1}^{i-1}\underline{B}_{ii}]})$$

and

$$\overline{\beta}_{ii} = 1/(\sqrt{[\sum_{i=0}^{m-1}(\overline{\psi}_i^{(i)}(0,r))^2 + \int_0^1 (\overline{\psi}_i^{(m)}(x,r))^2\mathrm{d}x - \sum_{i=1}^{i-1}\overline{B}_{ii}]})$$

$$(\underline{\beta}_{ij}, \overline{\beta}_{ij}) = (\underline{\beta}_{ii} * (-\sum_{k=j}^{i-1}\underline{B}_{ik} * \underline{\beta}_{kj}), \overline{\beta}_{ii} * (-\sum_{k=j}^{i-1}\overline{B}_{ik} * \overline{\beta}_{kj}))$$

$$(i = 1,2,\ldots), \qquad (j = 1,2,\ldots,i-1), \qquad (k = 1,2,\ldots,i-1).$$

4. Homotopy perturbation method

To illustrate the homotopy perturbation method (HPM) for solving non-linear differential equations, He [23, 24] considered the following non-linear differential equation:

$$A(u) = F(r);\ r \in \Omega, \tag{4.1}$$

subject to the boundary condition

$$B(u, \frac{\partial u}{\partial n}) = 0;\ r \in \Gamma, \tag{4.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω and $,\frac{\partial}{\partial n}$ denotes differentiation along the normal vector drawn outwards from Ω. The operator A can generally be divided into two parts M and N therefore, (1) can be rewritten as follows:

$$M(u) + N(u) = F(r);\ r \in \Omega. \tag{4.3}$$

He [23, 24] constructed a homotopy $v(r, p) : \Omega \times [0, 1] \to R$ which satisfies

$$H(v, p) = (1 - p)[M(v) - M(u_0)] + p[A(v) - f(r)] = 0. \tag{4.4}$$

Which is equivalent to

$$H(v, p) = M(v) - M(u_0) + pM(u_0) + p[N(v) - f(r)] = 0, \tag{4.5}$$

where $p \in [0, 1]$ is an embedding parameter, and u_0 is an initial approximation of (1). Obviously, we have

$$H(v, 0) = M(v) - M(u_0) = 0;\ H(v, 1) = A(v) - F(r) = 0. \tag{4.6}$$

The changing process of p from zero to unity is just that of $H(v, p)$ from $M(v) - M(u_0)$ to $A(v) - F(r)$. In topology, this is called deformation and $M(v) - M(u_0)$ and $A(v) - F(r)$ are called homotopic. According to the homotopy Perturbation method, the parameter p is used as a small parameter, and the solution of Eq.(4) can be expressed as a series in p in the form

$$u = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots . \tag{4.7}$$

When $p \to 1$, Eq. (4) corresponds to the original one. Eq. (7) becomes the approximate solution of Eq. (3), i.e.,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots , \tag{4.8}$$

The combination of perturbation method and homotopy method is called the HPM, which has eliminated the limitations of traditional perturbation methods. On the other hand, this technique is of full advantage of traditional perturbation techniques. Series (2.9) is convergent in most cases. However, the convergent rate depends on the nonlinear operator $A(V)$ (the following opinions are suggested by He [19])

(1) The second derivative of $N(V)$ with respect to V must be small because the parameter may be relatively large, i.e.,$p \to 1$.

(2) The norm of $L^{-1}(\frac{\delta N}{\delta V})$ must be smaller than one so that the series converges.

5. The application of HPM and FRKHSM to Solveing(1.1)

Analysis:Consider the fuzzy differential equation problem including Lane - Emden - type equation of form

$$\begin{cases} u^{''}(x) + \frac{k_1}{a(x)}u^{'}(x) + \frac{k_2}{b(x)}u(x) + f(x,u) = g(x), \quad 0 \lessdot x \leqslant 1, \\ u(0) = \alpha, u^{\cdot}(0) = \beta \end{cases} \tag{5.1}$$

The solution of equation can be expressed in following function:

$$Lu(x) = f(x)$$

Where

$$Lu(x) = a(x)b(x)u^{''}(x) + k_1 b(x)u^{'}(x) + k_2 a(x)u(x), f(x) = a(x)b(x)g(x)$$

If the crisp function $u(x), f(x)$ is continuous in the metric D,it is definite function exists. Furthermore

$$L\underline{u}(x,r) = \underline{f}(x,r),$$

and

$$L\bar{u}(x,r) = \bar{f}(x,r).$$

It shoud be noted that the fuzzy function cac be also defined using the Deravitive approach. Generally the integral of Eq.(x)and Eq.(y) are complicated and can not be expressed in term of elementary functions nor conveniently tabulated in open literature. However, this method is powerful tool to calculate such difficult fuzzy equations.We construcr the following homotopy with

$$\underline{M}(x,r) = k_2\underline{a}(x,r)\underline{u}(x,r),$$
$$\bar{M}(x,r) = k_2\bar{a}(x,r)\bar{u}(x,r),$$

and

$$\underline{N}(x,r) = \underline{a}(x,r)\underline{b}(x,r)\underline{u}''(x,r)$$
$$\bar{N}(x,r) = \bar{a}(x,r)\bar{b}(x,r)\bar{u}''(x,r),$$

$$(5.2)\qquad \begin{cases} \underline{M}(u,r) - \underline{M}(u_0,r) + p\underline{M}(u_0,r) + p[\underline{N}(v_0,r) - \underline{f}(x,r)] = 0, \\ \bar{M}(u,r) - \bar{M}(u_0,r) + p\bar{M}(u_0,r) + p[\bar{N}(v_0,r) - \bar{f}(x,r) = 0. \end{cases}$$

Substituting (7) into (12) . And equating cofficients of like power of p, we obtain:

$$(5.3)\qquad \begin{cases} \underline{M}(v_0,r) - \underline{M}(u_0,r) = \underline{a}(x,r)\underline{b}(x,r)\underline{u}_0''(x,r) + k_1\underline{b}(x,r)\underline{u}_0'(x,r) + & \\ k_2\underline{a}(x,r)\underline{u}_0(x,r) - \underline{g}(x,r) = 0, & \underline{u_0}(0,r) = \underline{\alpha}, \underline{u_0}'(0,r) = \underline{\beta}, \\ \bar{M}(v_0,r) - \bar{M}(u_0,r) = \bar{a}(x,r)\bar{b}(x,r)\bar{u}_0''(x,r) + k_1\bar{b}(x,r)\bar{u}_0'(x,r) + & \\ k_2\bar{a}(x,r)\bar{u_0}(x,r) - \bar{g}(x,r) = 0, & \bar{u_0}(0,r) = \bar{\alpha}, \bar{u}_0'(0,r) = \bar{\beta}, \end{cases}$$

$$(5.4)\qquad \begin{cases} \underline{M}(v_1,r) + \underline{M}(u_0,r) + \underline{N}(v_0,r) - \underline{a}(x,r)\underline{b}(x,r)\underline{g}(x,r) & \\ = \underline{a}(x,r)\underline{b}(x,r)\underline{u_1}''(x,r) + k_1\underline{b}(x,r)\underline{u_1}'(x,r) + k_2\underline{a}(x,r)\underline{u_1}(x,r) + & \\ \underline{a}((x,r)(u,r))\underline{b}((x,r)(u,r))\underline{g}((x,r)(u,r)) = 0, & \underline{u_1}(0,r) = 0, \underline{u_1}'(0,r) = 0, \\ \bar{M}(v_1,r) - \bar{M}(u_0,r) + \bar{N}(v_0,r) - \bar{a}(x,r)\bar{b}(x,r)\bar{g}(x,r) = & \\ \bar{a}(x,r)\bar{b}(x,r)\bar{u}_0''(x,r) + k_1\bar{b}(x,r)\bar{u}_1'(x,r) + k_2\bar{a}(x,r)\bar{u_1}(x,r) + & \\ \bar{a}((x,r)(u,r))\bar{b}((x,r)(u,r))\bar{g}((x,r)(u,r)) = 0, & \bar{u_1}(0,r) = 0, \bar{u}_1'(0,r) = 0 \end{cases}$$

$$(5.5)\qquad \begin{cases} \underline{M}(v_2,r) + \underline{N}(v_1,r) = \underline{a}(x,r)\underline{b}(x,r)\underline{u_2}''(x,r) + k_1\underline{b}(x,r)\underline{u_2}'(x,r) + k_2\underline{a}(x,r)\underline{u_2}(x,r) + & \\ \frac{d\underline{a}((x,r)(u,r))\underline{b}((x,r)(u,r))\underline{g}((x,r)(u,r))}{dp} = 0, & \underline{u_2}(0,r) = 0, \underline{u_2}'(0,r) = 0, \\ \bar{M}(v_2,r) + \bar{N}(v_1,r) = \bar{a}(x,r)\bar{b}(x,r)\bar{u}_2''(x,r) + k_1\bar{b}(x,r)\bar{u}_2'(x,r) + k_2\bar{a}(x,r)\bar{u_2}(x,r) + & \\ \frac{d\bar{a}((x,r)(u,r))\bar{b}((x,r)(u,r))\bar{g}((x,r)(u,r))}{dp} = 0, & \bar{u_2}(0,r) = 0, \bar{u}_2'(0,r) = 0 \end{cases}$$

$\vdots$

(5.6)

$$\begin{cases} \underline{M}(v_m,r) + \underline{N}(v_{m-1},r) = \underline{a}(x,r)\underline{b}(x,r)\underline{u_m}''(x,r) + k_1\underline{b}(x,r)\underline{u_m}'(x,r) + k_2\underline{a}(x,r)\underline{u_m}(x,r) + & \\ \frac{d^{m-1}\underline{a}((x,r)(u,r))\underline{b}((x,r)(u,r))\underline{g}((x,r)(u,r))}{(m-1)!dp^{m-1}} = 0, & \underline{u_m}(0,r) = 0, \underline{u_m}'(0,r) = 0, \\ \bar{M}(v_m,r) + \bar{N}(v_{m-1},r) = \bar{a}(x,r)\bar{b}(x,r)\bar{u}_m''(x,r) + k_1\bar{b}(x,r)\bar{u}_m'(x,r) + k_2\bar{a}(x,r)\bar{u_m}(x,r) + & \\ \frac{d^{m-1}\bar{a}((x,r)(u,r))\bar{b}((x,r)(u,r))\bar{g}((x,r)(u,r))}{(m-1)!dp^{m-1}} = 0, & \bar{u_m}(0,r) = 0, \bar{u}_m'(0,r) = 0 \end{cases}$$

To solve the above equation , we use the FRKHSM presented in section x and obtain$\underline{u}_0, \underline{u}_1, \underline{u}_2, \ldots$ $\bar{u}_0, \bar{u}_1, \bar{u}_2, \ldots$

$$(5.7)\qquad \begin{cases} \underline{u}_0(x,r) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \underline{\beta}_{ik}\underline{f}_0(x_k,r)\underline{\overline{\psi}}_i(x,r) \\ \bar{u}_0(x,r) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \bar{\beta}_{ik}\bar{f}_0(x_k,r)\overline{\overline{\psi}}_i(x,r) \end{cases}$$

$$(5.8)\qquad \begin{cases} \underline{u}_1(x,r) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \underline{\beta}_{ik}\underline{f}_1(x_k,r)\underline{\overline{\psi}}_i(x,r) \\ \bar{u}_1(x,r) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \bar{\beta}_{ik}\bar{f}_1(x_k,r)\overline{\overline{\psi}}_i(x,r) \end{cases}$$

$$(5.9)\qquad \begin{cases} \underline{u}_2(x,r) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \underline{\beta}_{ik}\underline{f}_2(x_k,r)\underline{\overline{\psi}}_i(x,r) \\ \bar{u}_2(x,r) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \bar{\beta}_{ik}\bar{f}_2(x_k,r)\overline{\overline{\psi}}_i(x,r) \end{cases}$$

$$\vdots$$

$$(5.10)\qquad \begin{cases} \underline{u}_m(x,r) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \underline{\beta}_{ik}\underline{f}_m(x_k,r)\underline{\overline{\psi}}_i(x,r) \\ \bar{u}_m(x,r) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \bar{\beta}_{ik}\bar{f}_m(x_k,r)\overline{\overline{\psi}}_i(x,r) \end{cases}$$

where

$$\begin{aligned} \underline{f}_0(x_k,r) &= \underline{a}(x_k,r)\underline{b}(x_k,r)\underline{g}(x_k,r) - k_1\underline{\beta}\underline{b}(x_k,r) + k_2\underline{a}(x_k,r)(\underline{\alpha} + \underline{\beta}(x_k,r)), \\ \bar{f}_0(x_k,r) &= \bar{a}(x_k,r)\bar{b}(x_k,r)\bar{g}(x_k,r) - k_1\bar{\beta}\bar{b}(x_k,r) + k_2\bar{a}(x_k,r)(\bar{\alpha} + \bar{\beta}(x_k,r)), \\ \underline{f}_1(x_k,r) &= -\underline{a}(x_k,r)\underline{b}(x_k,r)\underline{f}((x_k,r),(u,r))|_{p=0}(x,r), \\ \bar{f}_1(x_k,r) &= -\bar{a}(x_k,r)\bar{b}(x_k,r)\bar{f}((x_k,r),(u,r))|_{p=0}(x,r), \\ &\vdots \\ \underline{f}_m(x_k,r) &= \underline{a}(x_k,r)\underline{b}(x_k,r)\frac{d^{m-1}\underline{f}((x_k,r),(u,r))}{(m-1)!dp^{m-1}}|_{p=0}(x,r), \\ \bar{f}_m(x_k,r) &= \bar{a}(x_k,r)\bar{b}(x_k,r)\frac{d^{m-1}\bar{f}((x_k,r),(u,r))}{(m-1)!dp^{m-1}}|_{p=0}(x,r). \end{aligned}$$

Therefore, the approximate solution of (1.1) and m-term approximation to this solution are obtained

$$(5.11)\qquad \begin{cases} \underline{U}(x,r) = \sum_{k=0}^{\infty} \underline{u}_k(x,r) \\ \bar{U}(x,r) = \sum_{k=0}^{\infty} \bar{u}_k(x,r) \end{cases}$$

Now, the approximate solution $\bar{U}_{m,n}(x,r)$,$\underset{-}{U}_{m,n}(x,r)$ can be obtained by the n-term intercept of the $\bar{u}_k(x,r)$,$\underset{-}{u}_k(x,r)$,$k = 0, 1, 2, \ldots$,and

$$
(5.12) \qquad \begin{cases} \underset{-}{U}_{m.n}(x,r) = \sum_{k=0}^{m-1} \sum_{i=1}^{n} \sum_{j=1}^{i} \underset{-}{\beta_{ij}} \underset{-}{f}_k(x_j,r)\underset{-}{\bar{\psi}_j}(x,r) \\ \bar{U}_{m.n}(x,r) = \sum_{k=0}^{m-1} \sum_{i=1}^{n} \sum_{j=1}^{i} \bar{\beta}_{ij} \bar{f}_k(x_j,r)\bar{\bar{\psi}}_j(x,r) \end{cases}
$$

6. Conclusion

In this paper, the combination of homotopy perturbation and fuzzy reproducing kernel Hilbert space methods was employed successfully for solving nonlinear singular initial value problems including generalized Lane− Emden−type equations. The numerical results show that the present method is an accurate and reliable analytical technique for nonlinear singular initial value problems. Moreover, the method is also effective for solving other nonlinear initial value problems.

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