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FROZEN STEEPEST DESCENT METHOD FOR NONLINEAR ILL-POSED HAMMERSTEIN TYPE OPERATOR EQUATIONS

M. SABARI

Department of Mathematical and Computational Sciences, NIT Karnataka, Mangaluru -575 025, India
Email: msabarijothi16@gmail.com

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Abstract. In this study, we consider an inverse free iterative method for approximating a solution of the nonlinear ill-posed Hammerstein type equation $KF(x) = y$. Our approach is to solve $Kz = y$ and then $F(x) = z$. We use Tikhonov regularization method for approximating the solutions of $Kz = y$ and Frozen steepest descent method for solution of $F(x) = z$. The adaptive parameter choice strategy of Pereverzev and Schock (2005) is used for choosing the regularization parameter.
Key words and phrases. Nonlinear ill-posed Hammerstein type operator; Tikhonov regularization; Steepest descent method; Balancing principle.

1. Introduction

In this paper, we considered the problem of approximating the solution $\hat{x}$ of the nonlinear ill-posed Hammerstein type equation

$$KF(x) = y \tag{1.1}$$

where $F : D(F) \subseteq X \to Z$ is a Fréchet differentiable nonlinear operator, $K : Z \to Y$ is a bounded linear operator and X, Y, Z are Hilbert spaces. Throughout this paper, $D(F)$ stands for the domain of $F, \langle ., . \rangle$ and $\|.\|$ stand for inner product and norm, respectively. Fréchet derivative of F is denoted by $\mathbf{F}'(.)$ and its adjoint by $F'(.)^*$. A typical example of the Hammerstein type equation (1.1) is

$$KF(x)(t) := \int_o^1 k(s,t)x^3(s)ds$$

where $K : L^2[0,1] \to L^2[0,1]$ is a bounded linear operator defined by

$$Kz(t) = \int_0^1 k(s,t)z(s)ds,$$

with kernel $k(s,t) \in L^2([0,1] \times [0,1])$ and $F : D(F) \subseteq L^2[0,1] \to L^2[0,1]$ is the nonlinear operator defined by

$$Fx(s) = x^3(s).$$

In general (1.1) is ill-posed in the sense that the solution need not depends continuously on the right-hand side data y. In [6], George studied an iterative Newton-Tikhonov regularization (NTR) method for approximating a x_0 -minimum norm solution $\hat{x}$ of (1.1), where $\hat{x}$ is called an x_0-minimum norm solution, if

$$\|\hat{x} - x_0\| = \min\{\|x - x_0\| : KF(x) = y, x \in D(F)\}.$$

Further in practice, only an approximation of y, say y^δ with $\|y - y^\delta\| \leq \delta$ are available. So one has to consider

$$KF(x) = y^\delta \tag{1.2}$$

instead of (1.1). As in [1, 6, 7, 8, 9, 10, 14], we approach the problem (1.2) by solving the equation

$$Kz = y^\delta \tag{1.3}$$

first and then

$$F(x) = z. \tag{1.4}$$

For approximating $\hat{x}$, iterative regularization method are studied by Argyros et.al [1], Argyros et.al [4], George [6], George and Nair[7], George and Kuhanandan [8], George and Shobha [10] and Shobha et.al [14]. Note that, in all these methods, one has to compute the inverse involving Fréchet derivative of F at each iterate x_k or at initial guess x_0.

In the present study, we apply Tikhonov regularization to solve the linear operator equation (1.3) and then we consider the inverse free iterative method to solve the non-linear operator equation (1.4). The method involves, Fréchet derivative of F only at x_0(see (3.2)).

The rest of the paper is organized as follows: Section 2 contains preliminaries, Section 3 contains convergence analysis of inverse free iterative method, Section 4 contains error bounds and source conditions and Section 5 contains finite dimensional realization of inverse free iterative method. Finally the paper ends with an academic example in Section 6.

2. Preliminaries

Denote by $B_r(x)$, $\overline{B}_r(x)$ the open and closed ball in X, respectively, with center $x \in X$ and of radius $r > 0$. The following assumption is used for obtaining the error estimate.

Assumption 2.1. *There exists a continuous, strictly monotonically increasing function* $\varphi : (0, a] \to (0, \infty)$ *with* $a \geq \|K\|^2$ *satisfying;*

- $$lim_{\lambda \to 0} \varphi(\lambda) = 0$$
- $$\sup_{\lambda \geq 0} \frac{\alpha \varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha), \qquad \forall \alpha \in (0, a].$$
- *there exists* $v \in X$ *with* $\|v\| \leq 1$ *such that*
$$F(\hat{x}) - F(x_0) = \varphi(K^*K)v.$$

Let

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1} K^*(y^\delta - KF(x_0)) + F(x_0). \tag{2.1}$$

It is known that (see (4.3) in [8]) under the Assumption 2.1

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq \varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}. \tag{2.2}$$

3. Convergence analysis

Let $\delta_0 > 0$, $a_0 > 0$ be some constants with $\delta_0^2 < a_0$ and $\|x_0 - \hat{x}\| \leq r$. Let $\delta \in (0, \delta_0]$ and $\alpha \in [\delta_0^2, a_0]$. Then as in [3], for $\alpha > 0$, one can prove that

$$F'(x_0)^*(F(x) - z_\alpha^\delta) + \frac{\alpha}{c}(x - x_0) = 0 \tag{3.1}$$

has a unique solution x_α^δ in $B_r(x_0)$ provided $0 < r < \frac{1}{2k_0}$. To obtain an approximation for x_α^δ, we consider the iteration defined for $n = 0, 1, 2, \cdots$ by

$$x_{n+1} = x_n - \beta[F'(x_0)^*(F(x_n) - z_\alpha^\delta) + \frac{\alpha}{c}(x_n - x_0)]. \tag{3.2}$$

We need the following assumption for the convergence analysis of (3.2).

Assumption 3.1.
(a) There exists a constant $k_0 > 0$ *such that for every* $x \in D(F)$ *and* $v \in X$, *there exists an element* $\Phi(x, x_0, v) \in X$ *satisfying*

$$[F'(x) - F'(x_0)]v = F'(x_0)\Phi(x, x_0, v), \ \ \|\Phi(x, x_0, v)\| \leq k_0 \|v\| \|x - x_0\|.$$

(b)

$$\forall x \in B_r(\hat{x}), \|F'(x)\| \leq m.$$

Further, let β, $q_{\alpha,\beta}$ be parameters such that

$$\beta \leq \frac{1}{m^2 + \frac{a_0}{c}} \tag{3.3}$$

and

$$q_{\alpha,\beta} = 1 - \frac{\alpha\beta}{c} + \frac{3\beta m^2 k_0}{2} r. \tag{3.4}$$

The main result of this paper is the following theorem.

THEOREM 3.2. *Let Assumption 3.1 holds and let (x_n) be as in (3.2) and $0 < r < \min\{\frac{1}{2k_0}, \frac{2\alpha}{3m^2k_0}\}$. Then for each $\delta \in (0, \delta_0]$ and $c \leq \alpha$. Then the sequence (x_n) is in $B_{2r}(x_0)$ and converges to x_α^δ as $n \to \infty$. Further,*

$$\|x_{n+1} - x_\alpha^\delta\| \leq q_{\alpha,\beta}^{n+1} \|x_0 - x_\alpha^\delta\|, \tag{3.5}$$

where $q_{\alpha,\beta}$ is as in (3.4).

Proof: Clearly, $x_0 \in \overline{B_{2r}(x_0)}$. Let $M_n := \int_0^1 F'(x_\alpha^\delta + t(x_n - x_\alpha^\delta))dt$. Since $x_\alpha^\delta \in B_r(x_0)$, M_0 is well defined. Assume that for some $n > 0, x_n \in B_{2r}(x_0)$ and M_n is well defined. Then, since x_α^δ satisfies the equation (3.1), we have

$$\begin{aligned}
x_{n+1} - x_\alpha^\delta &= x_n - x_\alpha^\delta - \beta \left[F'(x_0)^*(F(x_n) - F(x_\alpha^\delta)) + \frac{\alpha}{c}(x_n - x_\alpha^\delta)\right] \\
&= x_n - x_\alpha^\delta - \beta \left[F'(x_0)^* M_n + \frac{\alpha}{c} I\right](x_n - x_\alpha^\delta) \\
&= x_n - x_\alpha^\delta - \beta \left[F'(x_0)^*(M_n - F'(x_0))\right](x_n - x_\alpha^\delta) \\
&\quad -\beta \left[F'(x_0)^* F'(x_0) + \frac{\alpha}{c} I\right](x_n - x_\alpha^\delta) \\
&= \left[I - \beta\left(F'(x_0)^* F'(x_0) + \frac{\alpha}{c} I\right)\right](x_n - x_\alpha^\delta) \\
&\quad -\beta[F'(x_0)^*(M_n - F'(x_0))](x_n - x_\alpha^\delta). \qquad (3.6)
\end{aligned}$$

Using Assumptions 3.1, we have

$$\begin{aligned}
x_{n+1} - x_\alpha^\delta &= \left[I - \beta\left(F'(x_0)^* F'(x_0) + \frac{\alpha}{c} I\right)\right](x_n - x_\alpha^\delta) \\
&\quad -\beta F'(x_0)^* F'(x_0) \int_0^1 \Phi(x_\alpha^\delta + t(x_n - x_\alpha^\delta), x_0, x_n - x_\alpha^\delta)dt.
\end{aligned}$$

Now since $I - \beta\left(F'(x_0)^*F'(x_0) + \frac{\alpha}{c}I\right)$ is a positive self-adjoint operator,

$$\begin{aligned}
& \|I - \beta\left(F'(x_0)^*F'(x_0) + \frac{\alpha}{c}I\right)\| \\
= & \sup_{\|x\|=1} |\langle\left(I - \beta\left(F'(x_0)^*F'(x_0) + \frac{\alpha}{c}I\right)\right)x, x\rangle| \\
= & |\sup_{\|x\|=1} \left(1 - \beta\frac{\alpha}{c}\right)\langle x, x\rangle - \beta\langle F'(x_0)^*F'(x_0)x, x\rangle| \\
\leq & 1 - \frac{\alpha\beta}{c}. \qquad (3.7)
\end{aligned}$$

The last step follows from the relation

$$\begin{aligned}
\beta|\langle F'(x_0)^*F'(x_0)x, x\rangle| & \leq \beta\|F'(x_0)\|^2 \\
& \leq \beta m^2 \\
& \leq \frac{1}{m^2 + \frac{a_0}{c}}m^2 \\
& \leq \frac{1}{m^2 + \frac{\alpha}{c}}m^2 = 1 - \frac{\alpha/c}{m^2 + \alpha/c} \leq 1 - \frac{\alpha\beta}{c}.
\end{aligned}$$

Hence, by Assumption 3.1, we have

$$\begin{aligned}
\|x_{n+1} - x_\alpha^\delta\| & \leq \left(1 - \frac{\alpha\beta}{c}\right)\|x_n - x_\alpha^\delta\| \\
& \quad + \beta m^2 k_0 \int_0^1 ((1-t)\|x_\alpha^\delta - x_0\| + t\|x_n - x_0\|)dt\|x_n - x_\alpha^\delta\| \\
& \leq \left(1 - \frac{\alpha\beta}{c} + \beta\frac{3k_0m^2r}{2}\right)\|x_n - x_\alpha^\delta\| \\
& \leq q_{\alpha,\beta}\|x_n - x_\alpha^\delta\|. \qquad (3.8)
\end{aligned}$$

Since $q_{\alpha,\beta} < 1$, we have

$$\|x_{n+1} - x_\alpha^\delta\| < \|x_0 - x_\alpha^\delta\| \leq r$$

and

$$\|x_{n+1} - x_0\| \leq \|x_{n+1} - x_\alpha^\delta\| + \|x_0 - x_\alpha^\delta\| \leq 2r$$

i.e., $x_{n+1} \in B_{2r}(x_0)$. Also, for $0 \leq t \leq 1$,

$$\|x_\alpha^\delta + t(x_{n+1} - x_\alpha^\delta) - x_0\| = \|(1-t)(x_\alpha^\delta - x_0) + t(x_{n+1} - x_\alpha^\delta)\| < 2r.$$

Hence, $x_\alpha^\delta + t(x_{n+1} - x_\alpha^\delta) \in B_{2r}(x_0)$ and M_{n+1} is well defined. Thus, by induction x_n is well defined and remains in $B_{2r}(x_0)$ for each $n = 0, 1, 2, \cdots$. By letting $n \to \infty$ in (3.2), we obtain the convergence of x_n to x_α^δ. The estimate (3.5) now follows from (3.8).

□

4. Error bounds under source conditions

In this section, we need the following assumptions in addition to the earlier assumptions to obtain the error bound.

Assumption 4.1. *There exists a continuous, strictly monotonically increasing function* $\varphi_1 : (0, b] \to (0, \infty)$ *with* $b \geq \|F'(x_0)\|^2$ *satisfying;*

- $$lim_{\lambda \to 0} \varphi_1(\lambda) = 0$$
- $$\sup_{\lambda \geq 0} \frac{\alpha \varphi_1(\lambda)}{\lambda + \alpha} \leq \varphi_1(\alpha), \qquad \forall \alpha \in (0, b].$$
- *there exists* $v \in X$ *with* $\|v\| \leq 1$ *such that*

$$x_0 - \hat{x} = \varphi_1(F'(x_0)^* F'(x_0))v.$$

Assumption 4.2. *For each* $x \in B_r(x_0)$, *there exists a bounded linear operator* $G(x, x_0)$ *such that*

$$F'(x) = F'(x_0)G(x, x_0)$$

with $\|G(x, x_0)\| \leq k_1$

Let $k_1 < \frac{1-k_0 r}{1-c}$ and assume that $\varphi_1(\alpha) \leq \varphi(\alpha)$. Proof of the following Theorems 4.3, 4.4 and 4.5 are analogous to the proof of Theorems 3.14, 3.15 and 3.16 in [1].

THEOREM 4.3. *(cf.* [1], *Theorem 3.14) Let* x_α^δ *be the solution of (3.1) and Assumption 4.1 and Assumption 4.2 hold. Let* $0 < r < \min\left\{\frac{1}{2k_0}, \frac{2\alpha}{3cm^2 k_0}\right\}$ *and* $k_1 < \frac{1-k_0 r}{1-c}$. *Then*

$$\|x_\alpha^\delta - \hat{x}\| \leq \frac{\varphi_1(\alpha) + \|F(\hat{x}) - z_\alpha^\delta\|}{1 - k_0 r - (1-c)k_1}.$$

THEOREM 4.4. *(cf.* [1], *Theorem 3.15) Let* (x_n) *be as in (3.2). Assumption 2.1 hold and* $\varphi_1(\alpha) \leq \varphi(\alpha)$ *and assumptions in Theorem 4.3 and Theorem 3.2 hold. Then*

$$\|x_n - \hat{x}\| \leq q_{\alpha,\beta}^n r + K\left(2\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}\right)$$

where $K = \frac{1}{1-K_0 r-(1-c)k_1}$.

THEOREM 4.5. *(cf.* [1], *Theorem 3.16) Let* (x_n) *be as in (3.2) and assumptions in Theorem 4.4 holds. Let*

$$n_k = \min\left\{n : q_{\alpha,\beta}^n \leq \frac{\delta}{\sqrt{\alpha}}\right\}.$$

Then

$$\|x_{n_k} - \hat{x}\| = \bar{K}\left(\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}\right).$$

where $\bar{K} = \max\{2K, r + K\}$

5. Finite dimensional realization of method (1.3)

Let $V_1 \subseteq V_2 \subseteq V_3 \subseteq .....$ be a sequence of finite-dimensional subspaces of X with $\overline{U_{n\in N} V_n} = X$ and P_h is the orthogonal projection of X onto V_n. Let

$$\varepsilon_h := \|K(I - P_h)\|,$$

$$\tau_h := \|F'(x)(I - P_h)\|, \quad \forall x \in D(F).$$

Let $\{b_h : h > 0\}$ is such that $\lim_{h\to 0} \frac{\|(I-P_h)x_0\|}{b_h} = 0$, $\lim_{h\to 0} \frac{\|(I-P_h)F(x_0)\|}{b_h} = 0$ and $\lim_{h\to 0} b_h = 0$. We assume that $\varepsilon_h \to 0$ and $\tau_h \to 0$ as $h \to 0$. The above assumption is satisfied if, $P_h \to I$ point wise and if K and $F'(x)$ are compact operators. Further we assume that $\varepsilon_h < \varepsilon_0$, $\tau_h \leq \tau_0$, $b_h \leq b_0$.

In the discretized Tikhonov regularization method for solving equation (2.1), the solution of $z_\alpha^{h,\delta}$ of the equation

$$\left(P_h K^* K P_h + \frac{\alpha}{c} P_h\right)(z_\alpha^{h,\delta} - P_h F(x_0)) = P_h K^*[y^\delta - KF(x_0)] \tag{5.1}$$

is taken as an approximation for $F(\hat{x})$.

THEOREM 5.1. *(See* [9], *Theorem 2.4) Let $z_\alpha^{h,\delta}$ be as in (5.1). Further if $b_h \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$ and Assumption 2.1 holds. Then*

$$\|F(\hat{x}) - z_{\alpha,h}^\delta\| \leq C\left(\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\sqrt{\alpha}}\right). \tag{5.2}$$

where $C = \max\{mr, 1\} + 1$

5.1. A priori choice of the parameter.

Note that the estimate $\varphi(\frac{\alpha}{c}) + \frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$ in (5.2) is of optimal order for the choice $\alpha := \alpha(\delta, h)$ which satisfies $\varphi(\alpha(\delta, h)) = \frac{\delta+\varepsilon_h}{\sqrt{\alpha(\delta,h)}}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq a$. Then we have $\delta + \varepsilon_h = \sqrt{\alpha(\delta, h)}\varphi(\alpha(\delta, h)) = \psi(\varphi(\alpha(\delta, h)))$ and

$$\alpha(\delta, h) = \varphi^{-1}(\psi^{-1}(\delta + \varepsilon_h)).$$

So from (5.2) we have $\|F(\hat{x}) - z_\alpha^{h,\delta}\| \leq 2C\psi^{-1}(\delta + \varepsilon_h)$.

5.2. An adaptive choice of the parameter.

Let

$$D_N = \{\alpha_i = \mu^i \alpha_0 : i = 1, 2, \dots N, \mu > 1, \alpha_0 > 0\}$$

be the set of possible values of the parameter α.

Let

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta + \varepsilon_h}{\sqrt{\alpha_i}}\right\} < N, \tag{5.3}$$

$$k = \max\left\{i : \alpha_i \in D_N^+\right\} \tag{5.4}$$

where $D_N^+ = \left\{\alpha_i \in D_N : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| \leq \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}, j = 0, 1, 2, ..., i-1\right\}$.

THEOREM 5.2. *(cf. [9], Theorem 2.5) Let l be as in (5.3), k be as in (5.4) and $z_{\alpha_k}^{h,\delta}$ be as in (5.1) with $\alpha = \alpha_k$. Then $l \leq k$ and*

$$\|F(\hat{x}) - z_{\alpha_k}^{h,\delta}\| \leq C\left(2 + \frac{4\mu}{\mu - 1}\right)\mu\psi^{-1}(\delta + \varepsilon_h).$$

Proof: Analogous to the proof of Theorem 2.5 in [9].

The discretized version of (3.2) as

$$x_{n+1,\alpha_k}^{h,\delta} = x_{n,\alpha_k}^{h,\delta} - \beta P_h\left[F'(x_0)^*(F(x_{n,\alpha_k}^{h,\delta}) - z_\alpha^{h,\delta}) + \frac{\alpha_k}{c}(x_{n,\alpha_k}^{h,\delta} - x_0^{h,\delta})\right] \tag{5.5}$$

where $x_0^{h,\delta} =: P_h x_0$ and $c \leq \alpha_k$. Let

$$(\delta_0 + \varepsilon_0)^2 < \bar{a}_0.$$

It is known that for [9, Theorem 3.7.] under the Assumption 2.1

$$P_h F'(x_0)^*(FP_h(x) - z_\alpha^{h,\delta}) + \frac{\alpha_k}{c}P_h(x - x_0) = 0 \tag{5.6}$$

has a unique solution $x_{\alpha_k}^{h,\delta}$ in $B_r(x_0) \cap R(P_h)$ and the following Theorems hold.

THEOREM 5.3. *(cf. [9], Theorem 3.8) Suppose $x_{\alpha_k}^{h,\delta}$ is the solution of 5.6 and Assumption 2.1 and Theorem hold. In addition if $\tau_0 < 1$, then*

$$\|x_{\alpha_k}^{h,\delta} - x_{\alpha_k}^\delta\| \leq \frac{1}{1-\tau_0}\left(\frac{\delta + \varepsilon_h}{\sqrt{\alpha_k}}\right).$$

Proof: Proof is analogous to the proof of Theorem 3.8 in [9]. □

The proof of the following Theorem 5.4 is analogous to the proof of Theorem 3.2 in Section 3.

THEOREM 5.4. *Let $x_{n,\alpha_k}^{h,\delta}$ be as in (5.5) and let $0 < r < \min\left\{\frac{2\alpha}{3M^2ck_0}, \frac{1}{2k_0}\right\}$. Then for each $\delta \in (0, \delta_0]$, $\alpha_k \in ((\delta + \varepsilon_h)^2, \bar{a_0}], \varepsilon_h \leq \varepsilon_0$ the sequence $\{x_{n,\alpha_k}^{h,\delta}\}$ is in $B_{2r}(x_0) \cap R(P_h)$ and converges to $x_{\alpha_k}^{h,\delta}$ as $n \to \infty$. Further,*

$$\|x_{n+1,\alpha_k}^{h,\delta} - x_{\alpha_k}^{h,\delta}\| \leq q_{\alpha_k,\beta}^{n+1}\|P_h x_0 - x_{\alpha_k}^{h,\delta}\|, \tag{5.7}$$

where $q_{\alpha_k,\beta}$ is as in (3.4) with $\alpha = \alpha_k$.

THEOREM 5.5. *Let $x_{\alpha_k}^{h,\delta}$ be the solution of (5.6) and Assumption in Theorem 4.3,5.3 and 5.4 hold. If $\varphi_1(\alpha) \leq \varphi(\alpha)$, then*

$$\|x_{n,\alpha_k}^{h,\delta} - \hat{x}\| \leq q_{\alpha_k,\beta}^n r + \left(\left(K + \frac{1}{1-\tau_0}\right) + KC\left(2 + \frac{4\mu}{\mu-1}\right)\right)\mu\psi^{-1}(\delta + \varepsilon_h).$$

where $q_{\alpha_k,\beta}$ is as in (3.4) with $\alpha = \alpha_k$.

By combing the results in Theorem 5.4 and Theorem 5.5, we obtain the following Theorem.

THEOREM 5.6. *Let $x^{h,\delta}_{n,\alpha_k}$ be as in (5.5) and assumptions in Theorem 5.5 holds. Let*

$$n_k = \min\left\{n : q^n_{\alpha_k,\beta} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\right\}.$$

Then

$$\|x^{h,\delta}_{n_k,\alpha_k} - \hat{x}\| = O(\psi^{-1}(\delta+\varepsilon_h)).$$

5.3. **Algorithm.** The balancing algorithm associated with the choice of the parameter specified in this Section involves the following steps:

- For $i, j \in \{0, 1, 2, \ldots, N\}$

$$z^\delta_{\alpha_i} - z^\delta_{\alpha_j} = (\alpha_j - \alpha_i)(K^*K + \alpha_i I)^{-1}[K^*(y^\delta - KF(x_0))];$$

- Choose $\alpha_0 = (\delta + \varepsilon_h)^2$ and $\mu > 1$;
- Choose $\alpha_i := \mu^{2i}\alpha_0, i = 0, 1, 2, \cdots, N$;
- Solve for $w_i : (K^*K + \alpha_i I)w_i = K * (y^\delta - KF(x_0)$;
- Solve for $j < i, z_{ij} : (K^*K + \alpha_i I)z_{ij} = (\alpha_j - \alpha_i)w_i$;
- $\|z_{ij}\| > 4C\frac{(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}$, then take k=i-1;
- Otherwise repeat with i+1 in place of i;
- Choose $n_k := min\left\{n : q^n_{\alpha_k,\beta} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\right\}$;
- Solve $x_k := x^{h,\delta}_{n_k,\alpha_k}$ by using the iteration (5.5).

6. Numerical Example

We consider the space $X = Y = L^2(0, 1)$ and the operator $KF : X \to Y$, where $F : D(F) \subseteq X \to Y$ is a nonlinear operator defined by

$$F(u) = \int_0^1 k(t, s)u^3(s)ds$$

and $K : X \to Y$ is a bounded linear operator defined by

$$K(x)(t) = \int_0^1 k(t, s)x(s)ds.$$

Here

$$k(t, s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1 \\ (1-s)t & 0 \leq s \leq t \leq 1 \end{cases}$$

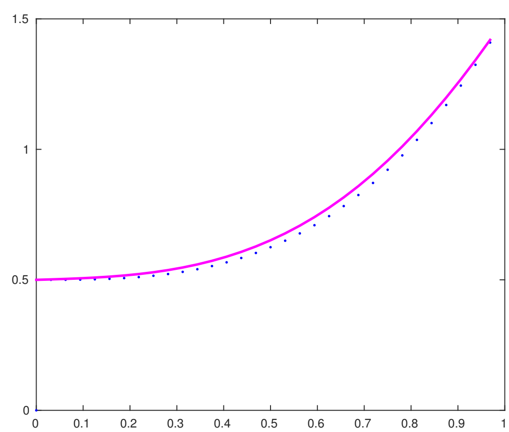
The Fréchet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t, s)u^2(s)w(s).0ds$$

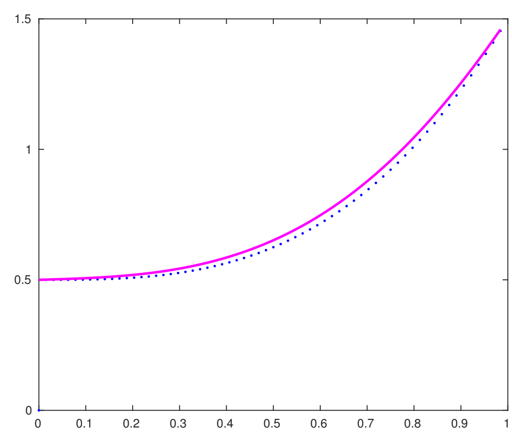
We have taken exact solution $\hat{x}(t) = 0.5 + t^3$ and initial guess $x_0(t) = 0.5 + t^3 - \frac{3}{56}(t - t^8)$. Let $\beta = 1/1000$. Then the error estimates are given in Table 1 and approximate and exact solutions for various values of δ are given in figures (a) to (f).

N	k	α_k	$\|x^{h,\delta}_{n_k,\alpha_k} - \hat{x}\|$	$\frac{\|x^{h,\delta}_{n_k,\alpha_k} - \hat{x}\|}{\sqrt{\delta+\varepsilon_h}}$
32	4	1.06E-01	2.36E-02	7.47E-02
62	4	1.06E-01	1.67E-02	5.28E-02
124	4	1.06E-01	1.18E-02	3.74E-02
256	4	1.06E-01	8.35E-03	2.64E-02
512	4	1.06E-01	5.91E-03	1.87E-02
1024	4	1.06E-01	4.18E-03	1.32E-02

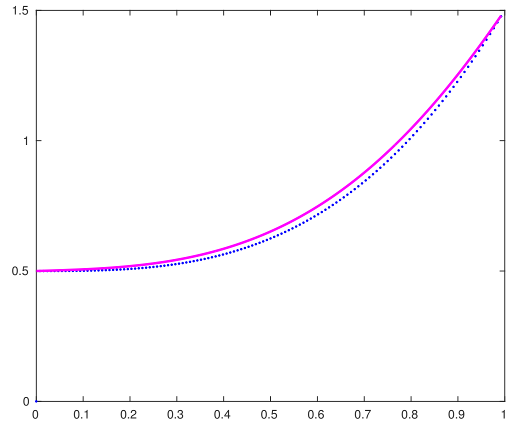
TABLE 1. Error estimate



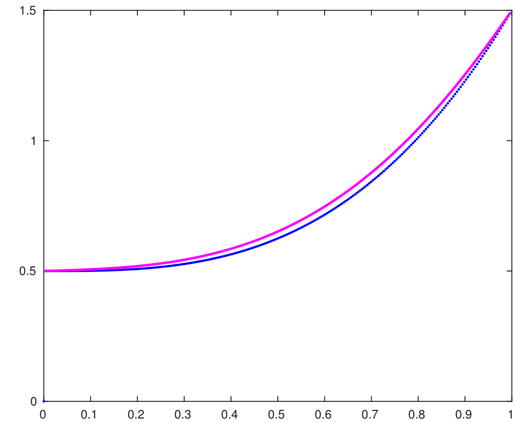
(a) $N = 32$



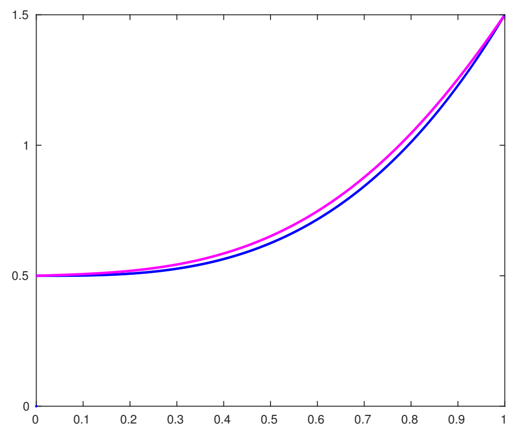
(b) $N = 64$



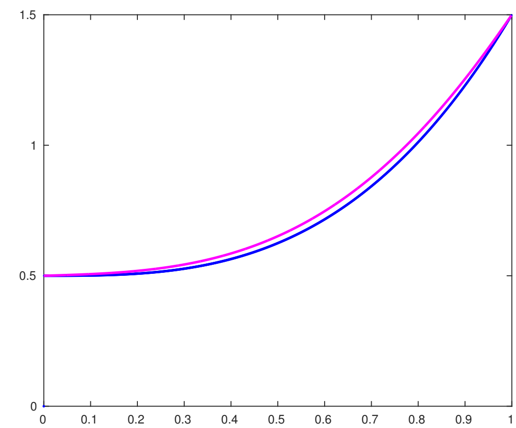
(c) $N = 128$



(d) $N = 256$



(e) $N = 512$



(f) $N = 1024$

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