Mathematical Inverse Problems, Vol. 4, No. 1 (2017), 1-11 ISSN 2381-9634

ERROR ESTIMATE FOR MODIFIED STEEPEST DESCENT METHOD FOR NONLINEAR ILL-POSED PROBLEMS UNDER HÖLDER-TYPE SOURCE CONDITION

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Received Mar 17, 2017

Abstract. Convergence rate result for steepest descent method for nonlinear ill-posed problems, under general Hölder-type source condition is not known. We consider a modified steepest descent method and obtained error estimate under general Hölder-type source condition. Discrepancy principle for modified steepest descent method with noisy data is also considered in this study. Numerical example is given to show the applicability of the modified method.
Key words and phrases. nonlinear ill-posed problem; steepest descent method; regularization method; discrepancy principle.

1. Introduction

Steepest descent type method is one of the iterative method used for approximately solving the nonlinear ill-posed operator equation

$$F(x) = y \tag{1.1}$$

when the exact data y is available. Here $F : D(F) \subseteq X \to Y$ is a nonlinear Fréchect differentiable operator between the Hilbert spaces X and Y and $D(F)$ denote the domain of F. We assumed that the operator equation (1.1) has a solution $\hat{x}$ for the exact data y and that we have only approximate data $y^\delta \in Y$ with

$$\|y - y^\delta\| \leq \delta.$$

The operator equation (1.1) is ill-posed in the sense that the solution $\hat{x}$ does not depend continuously on the right hand side data y (see [1–4] and reference therein). For exact data y, steepest descent method was studied by Neubauer and Scherzer in [9] and they obtained the convergence rate result under the source condition

$$x_0 - \hat{x} = (F'(\hat{x})^* F'(\hat{x}))^{\frac{1}{2}} v$$

for some $v \in X$. In [6], the authors studied the following modified steepest descent method

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \ \ (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_0)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|s_k\|^2}{\|F'(x_0)s_k\|^2} \end{aligned}$$

and obtained the convergence rate result under the source condition

$$x_0 - \hat{x} = (F'(x_0)^* F'(x_0))^{\frac{1}{2}} v$$

for some $v \in X$. But in the literature no convergence rate result is available under the general Hölder-type source condition

$$x_0 - \hat{x} = (F'(\hat{x})^* F'(\hat{x}))^{\nu} v \tag{1.2}$$

or

$$x_0 - \hat{x} = (F'(x_0)^* F'(x_0))^{\nu} v \tag{1.3}$$

for $\nu \neq \frac{1}{2}$. To obtain the convergence rate result under (1.3), we considered a new modified form of steepest descent method defined for $k = 0, 1, 2, \ldots$ by

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \\ s_k &= -F'(x_0)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|s_k\|^2}{\|A^q s_k\|^2} \end{aligned} \tag{1.4}$$

where $A = F'(x_0)^* F'(x_0)$ and $0 < q < \frac{1}{2}$. We need the following assumptions $(\mathcal{A})$:

- $(\mathcal{A}_0)$ $\|F'(x)\| \leq m$ for some $m > 0$ and for all $x \in D(F)$.
- $(\mathcal{A}_1)$ $F'(x) = R(x, y)F'(y)$ $(x, y \in B(x_0, \rho))$ where $\{R(x, y) : x, y \in B(x_0, \rho)\}$ is a family of bounded linear operators $R(x, y) : Y \longrightarrow Y$ with

  $$\|R(x, y) - I\| \leq C\|x - y\|$$

  for some positive constant C.

We obtained the error estimate $\|x_k - \hat{x}\| = O(k^{-\nu})$, for $0 < 2\nu < \frac{1}{2} - q, 0 < q < \frac{1}{2}$ under the assumption (1.3)(see Theorem 2.3). For noisy data y^δ, steepest descent method was studied by Scherzer in [10]. But no convergence rate result was available in [10]. We considered the method (1.4) with noisy data y^δ and obtained error estimate as in [6].

The rest of the paper is structured as follows. Convergence analysis of method (1.4) is given in Section 2 and Convergence rate result of method (1.4) with noisy data is given in Section 3. Finally, the paper ends with an example in Section 4.

2. Convergence analysis of method (1.4)

Our analysis in this section is based on the following result in [5](see [5, Lemma 2]). Let $\{v_k\}$ be a sequence in $X, \nu > 0$, be some parameter such that

$$\|A^\nu v_k\|^2 - \|A^\nu v_{k+1}\|^2 \geq \varepsilon_k \langle A^{\nu+1} v_k, A^\nu v_k \rangle$$

for $k = 0, 1, 2, \ldots$, where A is a positive self adjoint operator and $\varepsilon_k > 0$. Then

$$\|A^\nu v_k\| \leq [2(\nu+1)]^\nu \|v_k\|^{\frac{1}{\nu+1}} \left[\sum_{i=0}^{k-1} \varepsilon_i \|v_i\|^{-\frac{1}{\nu+1}} \right]^{-\nu}. \tag{2.1}$$

We shall apply the above result to $v_k = A^{-\nu}(x_k - \hat{x})$. Therefore, in order to apply (2.1), we need to prove;

$$\|x_k - \hat{x}\|^2 - \|x_{k+1} - \hat{x}\|^2 \geq \varepsilon_k \langle A(x_k - \hat{x}), x_k - \hat{x} \rangle \tag{2.2}$$

for some $\varepsilon_k > 0$ and $\|A^{-\nu}(x_k - \hat{x})\|$ is bounded. Let $B = \|A^{\frac{1}{2}-q}\|$ and $D = \frac{\sqrt{1+4B^2} - (B^2+1)}{B^2}$.

LEMMA 2.1. *Let the assumption ($\mathcal{A}_1$) and (1.3) hold with $0 < 2\nu < \frac{1}{2} - q, 0 < q < \frac{1}{2}$ and let $0 < C\rho < D$. Let x_k be as in (1.4). Then, $x_k \in B(x_0, 2\rho)$ and*

$$\|x_{k+1} - \hat{x}\|^2 + \alpha_k \Gamma \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \leq \|x_k - \hat{x}\|^2 \tag{2.3}$$

with

$$\Gamma = 2 - (B^2 C^2 \rho^2 + 2(B^2+1)C\rho + B^2), \tag{2.4}$$

for all $k = 0, 1, 2, \ldots$. Moreover,

$$\sum_{k=0}^{\infty} \alpha_k \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 < \infty.$$

Proof. We shall prove the result using induction. Note that $x_0 \in B(x_0, 2\rho)$ and suppose $x_k \in B(x_0, 2\rho)$. Then using (1.4), we have

$$\begin{aligned}
& \|x_{k+1} - \hat{x}\|^2 - \|x_k - \hat{x}\|^2 \\
= & -2\alpha_k \langle x_k - \hat{x}, F'(x_0)^*(F(x_k) - y) \rangle + \alpha_k^2 \|F'(x_0)^*(F(x_k) - y)\|^2 \\
= & -2\alpha_k \langle x_k - \hat{x}, F'(x_0)^* \left[F(x_k) - F(\hat{x}) - F'(x_0)(x_k - \hat{x}) \right] \rangle \\
& + \alpha_k \left[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\langle x_k - \hat{x}, F'(x_0)^* F'(x_0)(x_k - \hat{x}) \rangle \right] \\
= & -2\alpha_k \langle F'(x_0)(x_k - \hat{x}), \int_0^1 \left(F'(\hat{x} + t(x_k - \hat{x})) - F'(x_0) \right) dt (x_k - \hat{x}) \rangle \\
& + \alpha_k \left[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \right]. \qquad (2.5)
\end{aligned}$$

So by assumption $(\mathcal{A}_1)$, we have

$$\begin{aligned}
& \|x_{k+1}-\hat{x}\|^2 - \|x_k-\hat{x}\|^2 \\
= & -2\alpha_k \langle F'(x_0)(x_k-\hat{x}), \int_0^1 [R(\hat{x}+t(x_k-\hat{x}),x_0)-I]\, dt F'(x_0)(x_k-\hat{x})\rangle \\
& +\alpha_k \left[\alpha_k \|F'(x_0)^*(F(x_k)-y)\|^2 - 2\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2\right] \\
\leq & \ 2\alpha_k \int_0^1 \|R(\hat{x}+t(x_k-\hat{x}),x_0)-I\|\|F'(x_0)(x_k-\hat{x})\|^2 dt \\
& +\alpha_k \left[\alpha_k \|F'(x_0)^*(F(x_k)-y)\|^2 - 2\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2\right] \\
\leq & \ 2\alpha_k C\|\hat{x}+t(x_k-\hat{x})-x_0\|\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2 \\
& +\alpha_k \left[\alpha_k \|F'(x_0)^*(F(x_k)-y)\|^2 - 2\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2\right]. \qquad (2.6)
\end{aligned}$$

Note that

$$\begin{aligned}
\alpha_k\|F'(x_0)^*(F(x_k)-y)\|^2 &= \frac{\langle A^q s_k, A^{-q}s_k\rangle^2}{\|A^q s_k\|^2} \\
&\leq \frac{\|A^q s_k\|^2\|A^{-q}s_k\|^2}{\|A^q s_k\|^2} \\
&\leq \|A^{\frac{1}{2}-q}\|^2\|F(x_k)-y\|^2 \\
&= \|A^{\frac{1}{2}-q}\|^2\|\int_0^1 F'(\hat{x}+t(x_k-\hat{x}))dt(x_k-\hat{x})\|^2.
\end{aligned}$$

By assumption $(\mathcal{A}_1)$, we have

$$\begin{aligned}
& \alpha_k\|F'(x_0)^*(F(x_k)-y)\|^2 \\
\leq & \ \|A^{\frac{1}{2}-q}\|^2\|\int_0^1 [R(\hat{x}+t(x_k-\hat{x}),x_0)-I+I]\, dt F'(x_0)(x_k-\hat{x})\|^2 \\
\leq & \ \|A^{\frac{1}{2}-q}\|^2 \left(C\|\hat{x}+t(x_k-\hat{x})-x_0\|+1\right)^2 \|F'(x_0)(x_k-\hat{x})\|^2 \\
\leq & \ B^2(C\rho+1)^2\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2. \qquad (2.7)
\end{aligned}$$

Therefore, by (2.6) and (2.7) we have

$$\|x_{k+1}-\hat{x}\|^2 - \|x_k-\hat{x}\|^2 \leq \left[B^2C^2\rho^2 + 2(B^2+1)C_1\rho + B^2 - 2\right]\alpha_k\|A^{\frac{1}{2}}(x_k-\hat{x})\|^2.$$

This completes the proof. □

Next we will prove the boundedness of $\|A^{-\nu}(x_k-\hat{x})\|$. Let $B_1 = \|A^{\frac{1}{2}-\nu-q}\|, 0 < 2\nu < \frac{1}{2}-q$ with $0 < q < \frac{1}{2}$.

LEMMA 2.2. *Let the assumption $(\mathcal{A}_1)$ and (1.3) hold with $0 < 2\nu < \frac{1}{2}-q, 0 < q < \frac{1}{2}$ and $0 < C\rho < D$. Let x_k be as in (1.4). Then, $\|A^{-\nu}(x_k-\hat{x})\|$ is bounded.*

Proof. By using (1.3), one can prove that $x_k - \hat{x} \in R(A^\nu)$ for all $k = 0, 1, 2, \ldots$. So, we can apply $A^{-\nu}$ to $x_{k+1} - \hat{x}$ and $x_k - \hat{x}$. Then, we have

$$\begin{aligned} & \|A^{-\nu}(x_{k+1} - \hat{x})\|^2 - \|A^{-\nu}(x_k - \hat{x})\|^2 \\ = & -2\alpha_k \langle A^{-\nu}(x_k - \hat{x}), A^{-\nu} F'(x_0)^*(F(x_k) - y)\rangle \\ & +\alpha_k^2 \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\|^2 \\ \leq & \ 2\alpha_k \|A^{-\nu}(x_k - \hat{x})\| \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\| \\ & +\alpha_k^2 \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\|^2. \end{aligned} \tag{2.8}$$

From (2.8), we have

$$\|A^{-\nu}(x_{k+1} - \hat{x})\| \leq \|A^{-\nu}(x_k - \hat{x})\| + \alpha_k \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\|. \tag{2.9}$$

By the definition of α_k, we have

$$\begin{aligned} \alpha_k \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\|^2 & \leq \frac{\|A^\nu\|^2 \|A^{-\nu} s_k\|^2}{\|A^q s_k\|^2} \|A^{-\nu} s_k\|^2 \\ & = \frac{\|A^\nu\|^2}{\|A^q s_k\|^2} \langle A^q s_k, A^{-2\nu - q} s_k\rangle^2 \\ & \leq \|A^\nu\|^2 \|A^{\frac{1}{2} - 2\nu - q}\|^2 \|F(x_k) - y\|^2. \end{aligned} \tag{2.10}$$

Using Assumption $(\mathcal{A}_1)$ in (2.10), we get

$$\begin{aligned} & \alpha_k \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\|^2 \\ = & \ \|A^{\frac{1}{2} - 2\nu - q}\|^2 \| \int_0^1 [R(\hat{x} + t(x_k - \hat{x}), x_0) - I + I] \, dt F'(x_0)(x_k - \hat{x})\|^2 \\ \leq & \ \|A^{\frac{1}{2} - 2\nu - q}\|^2 \left(C\|\hat{x} + t(x_k - \hat{x}) - x_0\| + 1\right)^2 \|F'(x_0)(x_k - \hat{x})\|^2 \\ \leq & \ B_1^2 (C\rho + 1)^2 \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2, \end{aligned} \tag{2.11}$$

i.e.,

$$\sqrt{\alpha_k} \|A^{-\nu} F'(x_0)^*(F(x_k) - y)\| \leq B_1 (C\rho + 1) \|A^{\frac{1}{2}}(x_k - \hat{x})\|. \tag{2.12}$$

Using (2.12) in (2.9), we have

$$\|A^{-\nu}(x_{k+1} - \hat{x})\| \leq \|A^{-\nu}(x_k - \hat{x})\| + \sqrt{\alpha_k} B_1 (C\rho + 1) \|A^{\frac{1}{2}}(x_k - \hat{x})\|. \tag{2.13}$$

Let $z_k = \|A^{-\nu}(x_k - \hat{x})\|$. Then by (2.13), we have

$$z_{k+1} \leq z_k + B_1 (C\rho + 1) \sqrt{\alpha_k} \|A^{\frac{1}{2}}(x_k - \hat{x})\|.$$

By induction

$$z_k \leq z_0 + B_1 (C\rho + 1) \sum_{i=0}^{k-1} \sqrt{\alpha_i} \|A^{\frac{1}{2}}(x_i - \hat{x})\|. \tag{2.14}$$

Since the series $\sum_{k=0}^{\infty} \alpha_k \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2$ is bounded, there exists $M > 0$ such that

$$\sum_{i=0}^{k-1} \sqrt{\alpha_i} \|A^{\frac{1}{2}}(x_i - \hat{x})\| \leq M. \tag{2.15}$$

So by (2.14) and (2.15), we have

$$z_k \leq z_0 + B_1(C\rho + 1)M.$$

Since $z_0 = \|A^{-\nu}(x_0 - \hat{x})\| = \|A^{-\nu} A^{\nu} v\| = \|v\|$,

$$z_k \leq \|v\| + B_1(C\rho + 1)M. \tag{2.16}$$

This completes the proof. □

THEOREM 2.3. *Let the assumption ($\mathcal{A}_1$) and (1.3) for $0 < 2\nu < \frac{1}{2} - q, 0 < q < \frac{1}{2}$ hold and let $0 < C\rho < D$. Let x_k be as in (1.4). Then*

$$\|x_k - \hat{x}\| \leq \tilde{C} k^{-\nu}$$

where $\tilde{C} = [2(\nu + 1)]^{\nu} \epsilon^{-\nu} \left(\|v\| + B_1(C\rho + 1)M\right).$

Proof. Note that $\alpha_k \geq \|A^q\|^{-2}$. Since ($\mathcal{A}_1$) and (1.3) for $0 < 2\nu < \frac{1}{2} - q$ hold and $C\rho < D$. Set $\epsilon_k := \epsilon = \Gamma \|A^q\|^{-2}$ where Γ is as in (2.4). Now Lemma 2.2 implies

$$\begin{aligned} \|x_k - \hat{x}\|^2 - \|x_{k+1} - \hat{x}\|^2 &\geq \Gamma \alpha_k \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \\ &\geq \Gamma \|A^q\|^{-2} \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \\ &= \epsilon \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \\ &= \epsilon \langle F'(x_0)^* F'(x_0)(x_k - \hat{x}), x_k - \hat{x} \rangle \\ &= \epsilon \langle A(x_k - \hat{x}), x_k - \hat{x} \rangle. \end{aligned}$$

Therefore by (2.1), we have

$$\begin{aligned} \|x_k - \hat{x}\| &\leq [2(\nu + 1)]^{\nu} \|A^{-\nu}(x_k - \hat{x})\|^{\frac{1}{\nu+1}} \left[\sum_{i=0}^{k-1} \epsilon_i \|A^{-\nu}(x_i - \hat{x})\|^{\frac{-1}{\nu+1}}\right]^{-\nu} \\ &\leq [2(\nu + 1)]^{\nu} z_k^{\frac{1}{\nu+1}} \epsilon^{-\nu} \left[\sum_{i=0}^{k-1} z_i^{-\frac{1}{\nu+1}}\right]^{-\nu}. \end{aligned} \tag{2.17}$$

So by (2.16) and (2.17), we have

$$\begin{aligned} \|x_k - \hat{x}\| &\leq [2(\nu + 1)]^{\nu} \epsilon^{-\nu} \left(\|v\| + B_1(C\rho + 1)M\right) k^{-\nu} \\ &\leq \tilde{C} k^{-\nu}. \end{aligned} \tag{2.18}$$

□

REMARK 2.4. *Note that as* $q \to 0$*, we have* $\nu \to \frac{1}{4}$*. So we obtain the error estimate* $\|x_k - \hat{x}\| = O(k^{-\nu})$ *for* $0 < \nu < \frac{1}{4}$ *under general Hölder-type source condition (1.3).*

3. Convergence rate result of method (1.4) with noisy data

To obtain the error estimate for steepest descent method with noisy data we need the following assumption in addition to the earlier assumptions. As in [7], we assume that: $(\mathcal{A}_2)$ F satisfies the local property

$$\|F(u) - F(v) - F'(x_0)(u - v)\| \leq \eta \|F(u) - F(v)\|, \tag{3.1}$$

for all $u, v \in B(x_0, \rho)$ with $\max\{\frac{1-B^2}{3}, 0\} < \eta < 1 - \frac{B^2}{2}$.

The proofs of the following Proposition 3.1, Lemma 3.2 and Theorem 3.3 are analogous to the proof of Proposition 3.1, Lemma 3.3 and Theorem 3.4 in [6].

PROPOSITION 3.1. *(cf. [6], Proposition 3.1) Let the assumption* $(\mathcal{A}_2)$ *hold. Let* x_k^δ *be as in (1.4) with* y *replaced by* y^δ*. Then,* $x_k^\delta \in B(x_0, 2\rho) \subset D(F)$ *for all* $k = 0, 1, 2, \ldots$ *and*

$$\|F(x_k^\delta) - y^\delta\| > \tau\delta \tag{3.2}$$

where

$$\tau > 2\frac{(1+\eta)}{2 - 2\eta - B^2} > 2. \tag{3.3}$$

Moreover for all $0 \leq k < k_*$ *with* τ *as in (3.3), then we have*

$$k_*(\tau\delta)^2 \leq \sum_{k=0}^{k_*-1} \|F(x_k^\delta) - y^\delta\|^2 \leq \frac{\tau\|F'(x_0)\|^2}{(2 - 2\eta - B^2)\tau - 2(1+\eta)}\|x_0 - \hat{x}\|^2. \tag{3.4}$$

LEMMA 3.2. *(cf. [6], Lemma 3.3) Let* $C\rho < \frac{2(\tau-2)}{\tau}$*. Then* $\delta \leq (1 - \frac{C}{2}\|x_k^\delta - \hat{x}\|)\|F'(\hat{x})(x_k^\delta - \hat{x})\|$ *for all* $0 < k \leq k_*$*.*

Let $\Omega := \|A^q\|^{-2}\left((2 - 2\eta - B^2) - 2\frac{(1+\eta)}{\tau}\right)$.

THEOREM 3.3. *(cf. [6], Theorem 3.4) Let the assumption* $(\mathcal{A})$ *hold and let* $C\rho < \min\left\{\frac{2(\tau-2)}{\tau}, \frac{2}{m\sqrt{\Omega}}, 1\right\}$*. Let* x_{k+1}^δ *be as in (1.4). Then for* $0 \leq k < k_*$*,*

$$\|x_{k+1}^\delta - \hat{x}\| = \begin{cases} O(q^{\frac{k+1}{2}}) & if \quad \delta < q^{k+1} \\ O(\delta^{\frac{1}{2}}) & if \quad q^{k+1} \leq \delta \end{cases} \tag{3.5}$$

where $q := \max\left\{1 - \frac{C^2\Omega}{4}\|F'(\hat{x})(x_i^\delta - \hat{x})\|^2 : i = 0, 1, 2, \ldots k\right\}$.

REMARK 3.4. *Note that for each i,*

$$\begin{aligned}\frac{C^2\Omega}{4}\|F'(\hat{x})(x_i^\delta-\hat{x})\|^2 &\leq \frac{C^2\Omega}{4}\|F'(\hat{x})\|^2\|(x_i^\delta-\hat{x})\|^2\\ &\leq \frac{C^2\Omega}{4}m^2\rho^2.\end{aligned}$$

Since $C\rho < \frac{2}{m\sqrt{\Omega}}$, for $i = 0, 1, 2, \ldots k$, $\frac{C^2\Omega}{4}\|F'(\hat{x})(x_i^\delta-\hat{x})\|^2 < 1$. Therefore $1-\frac{C^2\Omega}{4}\|F'(\hat{x})(x_i^\delta-\hat{x})\|^2 < 1$ which implies $q < 1$.

4. Example

In this section, we consider the following example to implement the method (1.4)(see [8])

EXAMPLE 4.1. *(cf. [8]) Consider a nonlinear operator equation $F : L^2[0,1] \to L^2[0,1]$ defined by*

$$F(x) := (\arctan(x))^2. \tag{4.6}$$

The Fréchet derivative of F is

$$F'(x)w = \frac{2arctan(x)}{1+x^2}w.$$

If $x(t)$ vanishes on a set of positive Lebesgue measure, then $F'(x)$ is not boundedly invertible. If $x \in C[0,1]$ vanishes even at one point t_0, then $F'(x)$ is not boundedly invertible in $L^2[0,1]$.

Note that

$$F'(x)w = R(x,x_0)F'(x_0)w$$

with

$$R(x,x_0) = \frac{1+x_0^2}{1+x^2}\frac{arctan(x)}{arctan(x_0)},$$

respectively. Further, for $x_0 \neq 0$,

$$\|R(x,x_0)-I\| \leq \left[\frac{1}{\|arctan(x_0)\|} + 2\max\{\|x\|,\|x_0\|\}\right]\|x-x_0\|.$$

That is, assumption $(\mathcal{A}_1)$ is satisfied. Let us take $\hat{x}(t) = t, t \in [0,1]$ and $y(t) = arctan(t)^2$. We have taken initial guess $x_0(t) = t/2$ and $q = \frac{1}{4}$. Therefore $\nu < \frac{1}{8}$. For nosie free case, error estimates are given in table 1 and approximate solutions are given in figure 1. For noisy data, we have taken $\tau = 2.1$ and the error estimates are given in table 2 with different values of δ. Approximate solutions are given in figure 2(a), figure 2(b), figure 2(c) and figure 2(d).

TABLE 1. Error estimate for the method (1.4) with exact data

k	$\|x_k - \hat{x}\|$	$\frac{\|x_k - \hat{x}\|}{k^{\frac{1}{8}}}$
10	1.2173E-02	9.1287E-03
20	6.3958E-03	4.3981E-03
30	3.3920E-03	2.2172E-03
40	1.7654E-03	1.1132E-03
50	9.0892E-04	5.5739E-04
60	4.6533E-04	2.7893E-04
70	2.3753E-04	1.3966E-04
80	1.2107E-04	7.0008E-05
90	6.1662E-05	3.5135E-05
100	3.1393E-05	1.7654E-05

TABLE 2. Error estimate for the method (1.4) with noisy data

δ	k	$\|x_k^\delta - \hat{x}\|$	$\frac{\|x_k^\delta - \hat{x}\|}{\delta^{\frac{1}{2}}}$
0.1	2	5.3985E-02	1.7072E-01
0.01	4	3.0498E-02	3.0498E-01
0.001	13	8.7840E-03	2.7778E-01
0.0001	48	8.6070E-04	8.6070E-02

ACKNOWLEDGEMENT

One of the authors, Ms. Sabari, thanks National Institute of Technology Karnataka, India, for the financial support.

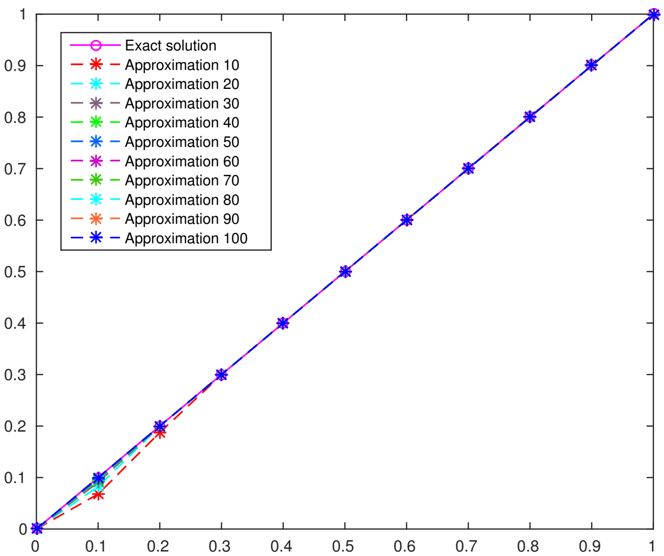

Figure 1. Approximate solutions for nosie free data

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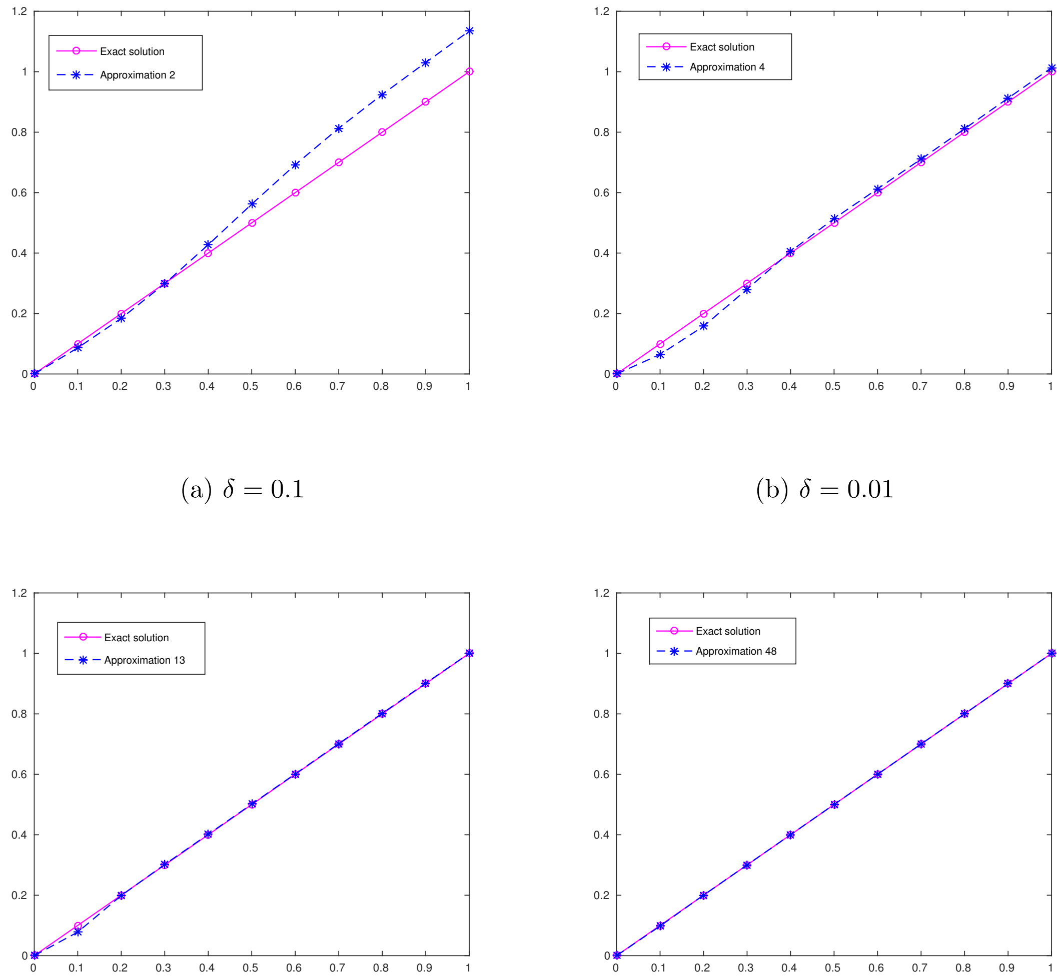

(a) $\delta = 0.1$

(b) $\delta = 0.01$

(c) $\delta = 0.001$

(d) $\delta = 0.0001$

FIGURE 2. Approximate solutions for noisy data