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HANKEL DETERMINANT FOR CERTAIN CLASSES OF (j,k)-SYMMETRIC FUNCTIONS

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Abstract. In the present investigation an upper bound of the second Hankel determinant $|a_2a_4 - a_3^2|$ for the functions belonging to the classes $\mathcal{S}^*_{j,k}$ and $\mathcal{K}_{j,k}$ are studied.
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1. **Introduction**

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C}\ and\ |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$. For f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$, if there exists an analytic function ω in $\mathcal{U}$ such that $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$, and we denote this by $f \prec g$. If g is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. The convolution or Hadamard product of two analytic functions $f, g \in \mathcal{A}$ where f is defined by (1) and $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$, then

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Definition 1.1. *Let k be a positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathcal{U}$ if for every z in $\mathcal{U}$*

$$f(e^{\frac{2\pi i}{k}} z) = e^{\frac{2\pi i}{k}} f(z).$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$ and for $k = 2$ we get class of the odd univalent functions.

The notion of (j,k)-symmetrical functions $(k = 2,3,\dots;j = 0,1,2,\dots,k-1)$ is a generalization of the notion of even, odd, k-symmetrical functions and also generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.
The theory of (j,k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan uniqueness theorem for holomorphic mappings [10].

Definition 1.2. *Let $\varepsilon = (e^{\frac{2\pi i}{k}})$ and $j = 0,1,2,..,k-1$ where $k \geq 2$ is a natural number. A function $f : \mathcal{U} \mapsto \mathbb{C}$ is called (j,k)-symmetrical if*

$$f(\varepsilon z) = \varepsilon^j f(z), \ z \in \mathcal{U}.$$

We note that the family of starlike functions with respect to (j,k)-symmetric points is denoted be $\mathcal{S}^{(j,k)}$. Also, $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are called even, odd and k-symmetric functions respectively. We have the following decomposition theorem.

Theorem 1.3. *[10] For every mapping $f : \mathcal{D} \mapsto \mathbb{C}$, and $\mathcal{D}$ is a k-fold symmetric set, there exists exactly the sequence of (j,k)- symmetrical functions $f_{j,k}$,*

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z)$$

where

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^v z), \tag{2}$$

$$(f \in \mathcal{A};\ k = 1,2,\dots;j = 0,1,2,\dots,k-1)$$

From (3) we can get

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^v z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj}\left(\sum_{n=1}^{\infty} a_n(\varepsilon^v z)^n\right),$$

then

$$f_{j,k}(z) = \sum_{n=1}^{\infty} \delta_{n,j} a_n z^n, \ a_1 = 1$$

$$\psi_n = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{(n-j)v} = \begin{cases} 1, & n = lk + j; \\ 0, & n \neq lk + j; \end{cases}, \tag{3}$$

We denote by $\mathcal{S}^*,\ \mathcal{K},\ \mathcal{S}_s^*,\ \mathcal{K}_s$ the familiar subclasses consisting of functions which, respectively, starlike, convex, starlike with respect to symmetric points and convex with respect to symmetric points in $\mathcal{U}$.

Definition 1.4. *A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}_{j,k}^*$ if*

$$\Re\left\{\frac{zf'(z)}{f_{j,k}(z)}\right\} > 0,$$

where $f_{j,k}$ defined by (2)*.*

Definition 1.5. *A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{K}_{j,k}$ if*

$$\Re\left\{\frac{(zf'(z))'}{f'_{j,k}(z)}\right\} > 0,$$

where $f_{j,k}$ defined by (2)*.*

In 1976, Noonan and Thomas [6] stated the q^{th} Hankel determinant of $f(z)$ for $q \geq 1$ and $n \geq 1$ as

$$\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}, \ a_1 = 1,$$

This determinant has also been considered by several authors. For example Noor in [7] determined the rate of growth $H_q(k)$ as $k \to \infty$ for functions f given by (1) with bounded boundary. Ehrenborg in [11] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [14].
Easily, one can observe that the Fekete and Szeg$\ddot{o}$ functional is $H_2(1)$. Fekete and Szeg$\ddot{o}$ [1] then further generalised the estimate $|a_3 - \mu a_2^2|$ where μ is real and $f \in \mathcal{S}$.

For our discussion in this paper, we consider the Hankel determinant tn the case of $q = 2$ and $n = 2$, known as second Hankel determinant:

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4 - a_3^2|,$$

and obtain an upper bound to $H_2(2)$ for $f(z) \in \mathcal{S}^{j,k}$ and $f(z) \in \mathcal{K}^{j,k}$. Janteng et al. [5] have considered the functional $|H_2(2)|$ and found a sharp bound, the subclass of $\mathcal{S}$ as $\Re\{f'(z)\} > 0$. In their work, they have shown that if $f \in \mathcal{S}$, then $|H_2(2)| \leq 4/9$. These authors [4, 8] also studied the second Hankel determinant and sharp bound for the classes of starlike and convex functions, close-to-starlike and close-to-convex functions with respect to symmetric points have shown that $|H_2(2)| \leq 1$, $|H_2(2)| \leq 1/8$, $|H_2(2)| \leq 1$, $|H_2(2)| \leq 1/9$, respectively.

2. Preliminary Results

Let $\mathcal{P}$ be the family of all functions p analytic in $\mathcal{U}$ for which $\Re\{p(z)\} > 0$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + ..., \; z \in \mathcal{U}$$

Lemma 2.1. *[9] If $p \in \mathcal{P}$, then $|p_n| \leq 2, (n = 1, 2, 3, ...)$.*

Lemma 2.2. *[12,13] If $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some x and z satisfying $|x| \leq 1, \; |z| \leq 1$ and $p_1 \in [0, 2]$.

3. Main Result

Theorem 3.1. *Let $f \in \mathcal{S}^*_{j,k}$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{4}{\delta_3^2}, \tag{4}$$

where $\delta_n = n - \psi_n$ and ψ_n defined by (3).

Proof. Since $f \in \mathcal{S}^*_{j,k}$, then there exist $p \in \mathcal{P}$ such that

$$\frac{zf'(z)}{f_{j,k}(z)} = p(z),$$

or

$$\frac{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}}{\sum_{n=1}^{\infty} \psi_n a_n z^{n-1}} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... \tag{5}$$

Equating coefficients in (5) yields

$$\psi_1 = 1, \; a_2 = \frac{p_1}{\delta_2}, \; a_3 = \frac{1}{\delta_3}\left[p_2 + \frac{\psi_2 p_1^2}{\delta_2}\right], \tag{6}$$

$$a_4 = \frac{1}{\delta_4}\left[p_3 + \frac{\psi_2 p_1 p_2}{\delta_2} + \frac{\psi_3 p_1}{\delta_3}\left(p_2 + \frac{\psi_2 p_1^2}{\delta_2}\right)\right]. \tag{7}$$

By (6), (7) we get

$$(8)\qquad |a_2a_4 - a_3^2| = \left| \frac{p_1}{\delta_2\delta_4}\left[p_3 + \frac{\psi_2 p_1 p_2}{\delta_2} + \frac{\psi_3 p_1}{\delta_3}\left(p_2 + \frac{\psi_2 p_1^2}{\delta_2}\right)\right] - \frac{1}{\delta_3^2}\left[p_2 + \frac{\psi_2 p_1^2}{\delta_2}\right]^2 \right|.$$

Using Lemma (2.1) and Lemma (2.2) in (8) we get

(9) $|a_2a_4 - a_3^2| =$

$$\left| \begin{array}{c} \frac{p_1}{4\delta_2\delta_4}\left[p_1^3 + 2p_1(4-p_1^2)x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)z\right] \\ +\frac{p_1}{\delta_2\delta_4}\left[\frac{\psi_2 p_1}{2\delta_2}\{p_1^2 + (4-p_1^2)x\} + \frac{\psi_3 p_1}{2\delta_3}\{p_1^2 + (4-p_1^2)x + \frac{2\psi_2 p_1^2}{\delta_2}\}\right] \\ -\frac{1}{\delta_3^2}\left[\frac{1}{4}\{p_1^4 + 2p_1^2(4-p_1^2)x + (4-p_1^2)^2x^2\} + (p_1^2 + (4-p_1^2)x)\frac{\psi_2 p_1^2}{\delta_2} + \frac{\psi_2^2 p_1^4}{\delta_2^2}\right] \end{array} \right|.$$

$$= \left| \begin{array}{c} \left[\frac{1}{2\delta_2\delta_4}\left\{1 + \frac{\psi_2}{\delta_2} + \frac{\psi_3}{\delta_3}\right\} - \frac{1}{2\delta_3^2} - \frac{\psi_2}{\delta_2\delta_3^2}\right] p_1^2(4-p_1^2)x \\ -\left[\frac{p_1^2}{4\delta_2\delta_4} + \frac{(4-p_1^2)}{4\delta_3^2}\right](4-p_1^2)x^2 + \frac{p_1(4-p_1^2)(1-|x|^2)z}{2\delta_2\delta_4} \\ +p_1^4\left[\frac{1}{\delta_2\delta_4}\left\{\frac{1}{4} + \frac{\psi_2}{2\delta_2} + \frac{\psi_3}{2\delta_3} + \frac{\psi_2\psi_3}{\delta_2\delta_3}\right\} - \frac{1}{4\delta_3^2} - \frac{\psi_2}{\delta_2\delta_3^2} - \frac{\psi_2^2}{\delta_2^2\delta_3^2}\right] \end{array} \right|.$$

Assume that $p_1 = p$ and $p \in [0,2]$, using triangular inequality and $|z| \le 1$, we have:

(10) $|a_2a_4 - a_3^2| \le$

$$\left\{ \begin{array}{c} \left[\frac{1}{2\delta_2\delta_4}\left\{1 + \frac{\psi_2}{\delta_2} + \frac{\psi_3}{\delta_3}\right\} + \frac{\psi_2}{\delta_2\delta_3^2} - \frac{1}{2\delta_3^2}\right] p^2(4-p^2)\mu \\ +\left[\frac{p^2}{4\delta_2\delta_4} + \frac{(4-p^2)}{4\delta_3^2}\right](4-p^2)\mu^2 + \frac{p(4-p^2)(1+\mu^2)}{2\delta_2\delta_4} \\ +p^4\left[\frac{1}{\delta_2\delta_4}\left\{\frac{1}{4} + \frac{\psi_2}{2\delta_2} + \frac{\psi_3}{2\delta_3} + \frac{\psi_2\psi_3}{\delta_2\delta_3}\right\} + \frac{\psi_2}{\delta_2\delta_3^2} - \frac{1}{4\delta_3^2} - \frac{\psi_2^2}{\delta_2^2\delta_3^2}\right] \end{array} \right\} = F(p,\mu).$$

Where $\mu = |x| \le 1$.

We next maximize the function $F(p,\mu)$ on the closed square $[0,2]\times[0,1]$. Differentiating $F(p,\mu)$ in (10) partially with respect to μ, we get.

$$(11)\quad \frac{\partial F(p,\mu)}{\partial \mu} = \left[\frac{p(p+2)}{4\delta_2\delta_4} + \frac{(4-p^2)}{4\delta_3^2}\right] 2(4-p^2)\mu + \left[\frac{1}{2\delta_2\delta_4}\left\{1 + \frac{\psi_2}{\delta_2} + \frac{\psi_3}{\delta_3}\right\} + \frac{\psi_2}{\delta_2\delta_3^2} - \frac{1}{2\delta_3^2}\right] p^2(4-p^2).$$

For $0 < \mu < 1$ and for fixed $0 < p < 2$, from (11), we observe that $\frac{\partial F(p,\mu)}{\partial \mu} > 0$. Consequently, $F(p,\mu)$ is increasing function of μ. Hence for fixed $p \in [0,2]$, the maximum of $F(p,\mu))$ occurs at $\mu = 1$, and

$$(12)\qquad \max_{0 \le \mu \le 1} F(p,\mu) = F(p,1) = G(p).$$

From the relations (10) and (12), upon simplification, we obtain

$$(13)\quad G(p) = F(p,1) = \left[\frac{\psi_2\psi_3}{\delta_2^2\delta_3\delta_4} + \frac{1}{2\delta_3^2} - \frac{\psi_2^2}{\delta_2^2\delta_3^2} - \frac{1}{2\delta_2\delta_4}\right] p^4 - p^3$$

$$+\left[\frac{2}{\delta_2\delta_4}\left\{1+\frac{\psi_2}{\delta_2}+\frac{\psi_3}{\delta_3}\right\}+\frac{4\psi_2}{\delta_2\delta_3^2}+\frac{1}{\delta_2\delta_4}-\frac{1}{\delta_3^2}\right]p^2+\frac{4}{\delta_2\delta_4}p+\frac{4}{\delta_3^2}.$$

Assume that $G(p)$ has a maximum value an interior of $p \in [0,2]$, by elementary calculation we find

$$G'(p)=4\left[\frac{\psi_2\psi_3}{\delta_2^2\delta_3\delta_4}+\frac{1}{2\delta_3^2}-\frac{\psi_2^2}{\delta_2^2\delta_3^2}-\frac{1}{2\delta_2\delta_4}\right]p^3-3p^2$$
$$+2\left[\frac{2}{\delta_2\delta_4}\left\{1+\frac{\psi_2}{\delta_2}+\frac{\psi_3}{\delta_3}\right\}+\frac{4\psi_2}{\delta_2\delta_3^2}+\frac{1}{\delta_2\delta_4}-\frac{1}{\delta_3^2}\right]p+\frac{4}{\delta_2\delta_4}.$$

Through some calculations we observe that all solutions of $G'(p)=0$ out of the interval $[0,2]$. A calculation showed that the maximum value occurs out of the interval which contradicts our assumption of having the maximum value at the interior point of $p \in [0,2]$. Thus any maximum point of G must be on the boundary of $p \in [0,2]$.
It is obvious that $G(0) > G(2)$. Hence G attains maximum value at $p = 0$. Therefore the upper bound for (10) corresponds to $\mu = 1$ and $p = 0$. Hence from (10) we obtain (4).

□

Setting $j = 1,\ k = 1$ in Theorem 4, we obtain the following result due to Janteng et al. [4].

Corollary 3.2. *If $f(z) \in \mathcal{S}^*$, then*

$$|a_2a_4-a_3^2| \le 1.$$

Setting $j = 1,\ k = 2$ in Theorem 4, we obtain the following result due to Janteng et al. [8].

Corollary 3.3. *If $f(z) \in \mathcal{S}_s^*$, then*

$$|a_2a_4-a_3^2| \le 1.$$

By using the similar method as in the proof of Theorem 4, one can similarly prove Theorem 3.4.

Theorem 3.4. *Let $f \in \mathcal{K}_{j,k}$, then*

(14) $$|a_2a_4-a_3^2| \le \frac{4}{9\delta_3^2}.$$

Where $\delta_n = n-\psi_n$ and ψ_n defined by (3).

Setting $j = 1,\ k = 1$ in Theorem 3.4, we obtain the following result due to Janteng et al. [4].

Corollary 3.5. *If $f(z) \in \mathcal{K}$, then*

$$|a_2a_4 - a_3^2| \leq \frac{1}{9}$$

Setting $j = 1,\ k = 2$ in Theorem 3.4, we obtain the following result due to Janteng et al. [8].

Corollary 3.6. *If $f(z) \in \mathcal{K}_s$, then*

$$|a_2a_4 - a_3^2| \leq \frac{1}{9}$$

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