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# NUMERICAL SOLUTION OF FUZZY VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND

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Abstract. In this paper, we use parametric form of fuzzy number and convert a fuzzy Volterra integral equation of the first kind with regular, as well as weakly singular k-ernels to a system of integral equations in crisp case. This paper presents a method based on polynomial approximation using polynomial basis to obtain approximate numerical solution of this system and hence obtain an approximation for fuzzy solution of the fuzzy Volterra integral equation of the first kind. This method using simple computation with quite acceptable approximate solution. However, accuracy and efficiency are dependent on the size of the set of Bernstein polynomials. Furthermore we get an estimation of error bound for this method.
Key words and phrases. Fuzzy integral equations; System of Volterra integral equation of the first kind with singular kernel; Bernstein polynomial.

## 1. Introduction

The solutions of integral equations have a major role in the field of science and engineering. A physical even can be modelled by the differential equation, an integral equation. Since few of these equations cannot be solved explicitly, it is often necessary to resort to numerical techniques which are appropriate combinations of numerical integration and interpolation [3, 15]. There are several numerical methods for solving linear Volterra integral equation [6, 21] and system of nonlinear Volterra integral equations [4]. Kauthen in [13] used a collocation method to solve the Volterra-Fredholm integral equation numerically. Maleknejad and et al. in [17] obtained a numerical solution of Volterra integral equations by using Bernstein Polynomials.

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [23], Dubois and Prade [8]. We refer the reader to [11] for more information on fuzzy numbers and fuzzy arithmetic. The topics of fuzzy integral equations (FIE) which growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in recent years. The fuzzy mapping function was introduced by Chang and Zadeh [5]. Later, Dubois and Prade [9] presented an elementary fuzzy calculus based on the extension principle also the concept of integration of fuzzy functions was first introduced by Dubois and Prade [9]. Babolian et al., Abbasbandy et al. in [2, 1] obtained a numerical solution of linear Fredholm fuzzy integral equations of the second kind.

In this paper, we present a novel and very simple numerical method based upon Bernstein's approximation for solving fuzzy Vlterra integral equation of the first kind with singular kernel.

## 2. Preliminaries

In this section the basic notations used in fuzzy calculus and Bernstein polynomials are introduced. We start by defining the fuzzy number.

**Definition 1.** [14] A fuzzy number is a fuzzy set $u : \mathbb{R}^1 \longrightarrow I = [0, 1]$ such that

i. $u$ is upper semi-continuous;

ii. $u(x) = 0$ outside some interval $[a, d]$;

iii. There are real numbers $b$ and $c$, $a \le b \le c \le d$, for which

1. $u(x)$ is monotonically increasing on $[a, b]$,
2. $u(x)$ is monotonically decreasing on $[c, d]$,
3. $u(x) = 1, b \le x \le c$.

The set of all the fuzzy numbers (as given in definition 1) is denoted by $E^1$.

An alternative definition which yields the same $E^1$ is given by Kaleva [12, 16].

**Definition 2.** A fuzzy number $u$ is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \le r \le 1$, which satisfy the following requirements:

i. $\underline{u}(r)$ is a bounded monotonically increasing, left continuous function on $(0, 1]$ and right continuous at $0$;

ii. $\overline{u}(r)$ is a bounded monotonically decreasing, left continuous function on $(0, 1]$ and right continuous at $0$;

iii. $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$.

A crisp number $r$ is simply represented by $\underline{u}(\alpha) = \overline{u}(\alpha) = r, 0 \le \alpha \le 1$. The set of all the fuzzy numbers is denoted by $E^1$.

For arbitrary $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and $k \in \mathbb{R}$ we define addition and multiplication by $k$ as

$$\begin{aligned}
&\underline{(u+v)}(r) = (\underline{u}(r) + \underline{v}(r)),\\
&\overline{(u+v)}(r) = (\overline{u}(r) + \overline{v}(r)),\\
&\underline{ku}(r) = k\underline{u}(r), \overline{ku}(r) = k\overline{u}(r),\ if\ k \ge 0,\\
&\underline{ku}(r) = k\overline{u}(r), \overline{ku}(r) = k\underline{u}(r),\ if\ k < 0.
\end{aligned}$$

**Remark 1.** [1] Let $u = (\underline{u}(r), \overline{u}(r)), 0 \le r \le 1$ be a fuzzy number, we take

$$u^c(r) = \frac{\underline{u}(r) + \overline{u}(r)}{2},$$

$$u^d(r) = \frac{\overline{u}(r) - \underline{u}(r)}{2}.$$

It is clear that $u^d(r) \ge 0$, $\underline{u}(r) = u^c(r) - u^d(r)$ and $\overline{u}(r) = u^c(r) + u^d(r)$, also a fuzzy number $u \in E^1$ is said symmetric if $u^c(r)$ is independent of $r$ for all $0 \le r \le 1$.

**Remark 2.** Let $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and also $k, s$ are arbitrary real numbers. If $w = ku + sv$ then

$$w^c(r) = ku^c(r) + sv^c(r),$$
$$w^d(r) = |k|u^d(r) + |s|v^d(r).$$

**Definition 3.** [10] For arbitrary fuzzy numbers $u, v$, we use the distance

$$D(u, v) = sup_{0 \le r \le 1} max\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\}$$

and it is shown that $(E^1, D)$ is a complete metric space [20].

The Bernstein's approximation, $B_n(f)$ to a real function $f : [0,1] \longrightarrow \mathbb{R}$ is the polynomial

$$B_n(f(x)) = \sum_{i=0}^{n} f(\frac{i}{n}) P_{n,i}(x), \tag{1}$$

where

$$P_{n,i}(x) = \left( \begin{array}{c} n \\ i \end{array} \right) x^i (1-x)^{n-i}, \quad i = 0, 1, ..., n.$$

There are $n+1$ $n$th-degree polynomials. For convenience, we set $P_{n,i}(x) = 0$, if $i < 0$ or $i > n$. It can be readily shown that each of the Bernstein polynomials is positive.

**Theorem 1.** For all functions $f$ in $C[0,1]$, the sequence $B_n(f); n = 1, 2, 3, ...$ converges uniformly to $f$, where $B_n$ is defined by Eq. (1).

**Proof.** See [19]. □

This theorem follows that, for any $f \in C[0,1]$ and for any $\epsilon > 0$, there exists $n$ such that the inequality $\|B_n(f) - f\| < \epsilon$, holds.

We suppose $\|.\|$ be the max norm on $[0,1]$, then the error bound

$$|B_n(f(x)) - f(x)| \le \frac{1}{2n} x(1-x) \|f''\|, \tag{2}$$

given in [7], shows that the rate of convergence is at least $\frac{1}{n}$ for $f \in C[0,1]$. On the other hand, the asymptotic formula

$$lim_{n \longrightarrow \infty} n(B_n(f(x)) - f(x)) = \frac{1}{2} x(1-x) f''(x), \tag{3}$$

due to Voronovskaya [22] shows that for $x \in (0,1)$ with $f' \neq 0$, the rate of convergence is precisely $\frac{1}{n}$.

## 3. Fuzzy Volterra integral equation of the first kind

The Fuzzy Volterra integral equations of the first kind (FVIE-1) is [18]

$$\lambda \int_0^x k(x,t)F(t)dt = G(x), \quad 0 \leq x \leq 1. \tag{4}$$

where $\lambda > 0, k(x,t)$ is a kernel function and $G(x)$ is a fuzzy function. If $G(x)$ is a fuzzy function these equation may only possess fuzzy solution. Sufficient conditions for the existence of a unique solution to the fuzzy Volterra integral equation are given in [18].

Now, we introduce parametric form of a FVIE-1 with respect to Definition 2. Let $(\underline{G}(x;r), \overline{G}(x;r))$ and $(\underline{F}(t;r), \overline{F}(t;r)),\ 0 \leq r \leq 1$ are parametric form of $G(x)$ and $F(t)$, respectively then, parametric form of FVIE-1 is as follows:

$$\lambda \int_0^x \underline{k(x,t)F(t;r)}dt = \underline{G}(x;r),\ \ 0 \leq x \leq 1,\ \ 0 \leq r \leq 1, \tag{5}$$

$$\lambda \int_0^x \overline{k(x,t)F(t;r)}dt = \overline{G}(x;r),\ \ 0 \leq x \leq 1,\ \ 0 \leq r \leq 1. \tag{6}$$

Suppose $k(x,t)$ be continuous on the square $[0,1]^2$ and for fix $t$, $k(x,t)$ changes its sign in finite points as $x_j$ where $x_j \in [0,x]$. For example, let $k(x,t)$ be nonnegative over $[0,x_1]$ and negative over $[x_1,x]$, therefore from Eqs. (5) and (6), we have

$$\begin{gathered}\lambda \int_0^{x_1} k(x,t)\underline{F(t;r)}dt + \lambda \int_{x_1}^{x} k(x,t)\overline{F(t;r)}dt = \underline{G}(x;r), \\ 0 \leq x \leq 1,\ \ 0 \leq r \leq 1,\end{gathered}$$

$$\begin{gathered}\lambda \int_0^{x_1} k(x,t)\overline{F(t;r)}dt + \lambda \int_{x_1}^{x} k(x,t)\underline{F(t;r)}dt = \overline{G}(x;r), \\ 0 \leq x \leq 1,\ \ 0 \leq r \leq 1.\end{gathered}$$

By referring to Remark 2 we have

$$\lambda \int_0^x k(x,t)F^c(t;r)dt = G^c(x;r),\ \ 0 \leq x \leq 1,\ \ 0 \leq r \leq 1, \tag{7}$$

$$\lambda \int_0^x |k(x,t)|F^d(t;r)dt = G^d(x;r),\ \ 0 \leq x \leq 1,\ \ 0 \leq r \leq 1. \tag{8}$$

It is clear that we must solve two crisp Volterra integral equation of the first kind provided that each of Eqs. (7) and (8) have solution.

## 4. Discretization of integral equations VK1 by using Bernstein's approximation

We consider the Volterra integral equations of the first kind given by,

$$\lambda \int_0^x k(x,t) F^c(t;r) dt = G^c(x;r), \quad 0 \leq x \leq 1, \quad 0 \leq r \leq 1,$$

$$\lambda \int_0^x |k(x,t)| F^d(t;r) dt = G^d(x;r), \quad 0 \leq x \leq 1, \quad 0 \leq r \leq 1,$$

where $F^c(t;r)$ and $F^d(t;r)$ are the unknown crisp function to be determined, $k(x,t)$ is a continuous function on the square $[0,1]^2$ and integrable function, $G^c(x;r)$ and $G^d(x;r)$ being the known crisp functions.

To determine an approximate the unknown function of Eq. (4), we approximate with Bernstein's approximation

(9) $$\lambda \int_0^x k(x,t) B_n(F(t)) dt = G(x), \quad 0 \leq x \leq 1,$$

therefore, we approximate the unknown functions $F^c(t;r)$ and $F^d(t;r)$ by

(10) $$B_n(F^c(t;r)) = \sum_{i=0}^{n} F^c(\frac{i}{n};r) P_{n,i}(t),$$

and

(11) $$B_n(F^d(t;r)) = \sum_{i=0}^{n} F^d(\frac{i}{n};r) P_{n,i}(t),$$

where

$$P_{n,i}(t) = \begin{pmatrix} n \\ i \end{pmatrix} t^i (1-t)^{n-i}, \quad i = 0, 1, ..., n.$$

Let $(\underline{B_n(F(t;r))}, \overline{B_n(F(t;r))}), \ 0 \leq r \leq 1$ is a parametric form of $B_n(F(t))$, then we have:

$$\underline{B_n(F(t;r))} = \sum_{i=0}^{n} \underline{F(\frac{i}{n};r)} P_{n,i}(t), \quad 0 \leq r \leq 1,$$

$$\overline{B_n(F(t;r))} = \sum_{i=0}^{n} \overline{F(\frac{i}{n};r)} P_{n,i}(t), \quad 0 \leq r \leq 1.$$

By referring to Remark 2, we have the following equations

(12) $$\lambda \int_0^x k(x,t) (\sum_{i=0}^n \begin{pmatrix} n \\ i \end{pmatrix} F^c(\tfrac{i}{n};r) t^i (1-t)^{n-i}) dt = G^c(x;r), \quad 0 \leq r \leq 1,$$

(13) $$\lambda \int_0^x |k(x,t)| (\sum_{i=0}^n \begin{pmatrix} n \\ i \end{pmatrix} F^d(\tfrac{i}{n};r) t^i (1-t)^{n-i}) dt = G^d(x;r), \quad 0 \leq r \leq 1.$$

In order to find $F^c(\frac{i}{n};r)$ and $F^d(\frac{i}{n};r)$ for $i=0,1,...,n$, we now put $x=x_j, j=0,1,...,n$ in (12) and (13), $x_j$'s being chosen as suitable distinct points in $(0,1)$, and $x_0$ is taken near $0$ and $x_n$ near $1$ such that $0<x_0<x_1<...<x_n<1$. Putting $x=x_j$ we obtain in short form two linear systems

(14) $$A_1X_1=Y_1,$$

where

$$A_1=[\lambda\begin{pmatrix} n \\ i \end{pmatrix}\int_0^{x_j}k(x_j,t)t^i(1-t)^{n-i}dt],$$
$$i,j=0,1,...,n,$$

$$X_1=[F^c(\tfrac{i}{n};r)]^t,\ \ 0\le r\le 1,\ i=0,1,...,n,$$

$$Y_1=[G^c(x_j;r)]^t,\ \ 0\le r\le 1,\ j=0,1,...,n,$$

and also

(15) $$A_2X_2=Y_2,$$

where

$$A_2=[\lambda\begin{pmatrix} n \\ i \end{pmatrix}\int_0^{x_j}|k(x_j,t)|t^i(1-t)^{n-i}dt],$$
$$i,j=0,1,...,n,$$

$$X_2=[F^d(\tfrac{i}{n};r)]^t,\ \ 0\le r\le 1,\ i=0,1,...,n,$$

$$Y_2=[G^d(x_j;r)]^t,\ \ 0\le r\le 1,\ j=0,1,...,n.$$

In general we cannot be able to carry out analytically the integrations, involved. We compute the integral that exist in $A_1's$ formula and $A_2's$ formula numerically. Now we can show $F^c(\frac{i}{n};r)$ and $F^d(\frac{i}{n};r)$ by $F^c_n(\frac{i}{n};r)$ and $F^d_n(\frac{i}{n};r),\ i=0,1,...,n,$ respectively that are our solutions in nodes $x_j, j=0,1,...,n$ and by substituting them in Eqs. (10) and (11) we can find $B_n(F^c_n(x_j;r))$ and $B_n(F^d_n(x_j;r))j=0,1,...,n$ that are solution for integral equations (7) and (8).

We give error bound for this solution in the following theorem.

**Theorem 2.** Consider the crisp Volterra integral equations of the first kind (7) and (8).

Assume that $k(x,t)$ is continuous on the square $[0,1]^2$ and the solution of the equations belong to $(C^\alpha \cap L^2)([0,1])$ for some $\alpha > 2$. If $A_1$ and $A_2$ invertible then

$$sup_{x_i \in [0,1]} D(F(x_i), B_n(F_n(x_i))) \leq$$

$$sup_{r \in [0,1]} [\tfrac{1}{8n}(M \|A_1^{-1}\| \|F''^c(r)\| + \|F_n''^c(r)\|) +$$

$$\tfrac{1}{8n}(M \|A_2^{-1}\| \|F''^d(r)\| + \|F_n''^d(r)\|)],$$

where $x_i = \frac{i}{n},\ i = 0,1,...,n,\ F(x)$ is exact solution of FVIE-1 and $M = sup_{x,t \in [0,1]} \mid \lambda k(x,t) \mid$.

**Proof.** We have

$$\begin{aligned} &sup_{x_i \in [0,1]} |F^\dagger(x_i;r) - B_n(F_n^\dagger(x_i;r))| = \\ &sup_{x_i \in [0,1]} |F^\dagger(x_i;r) - F_n^\dagger(x_i;r) + F_n^\dagger(x_i;r) - B_n(F_n^\dagger(x_i;r))| \leq \\ &sup_{x_i \in [0,1]} |F^\dagger(x_i;r) - F_n^\dagger(x_i;r)| + sup_{x_i \in [0,1]} |F_n^\dagger(x_i;r) - B_n(F_n^\dagger(x_i;r))|, \end{aligned} \tag{16}$$

where $\dagger$ means we have this equation for $c$ and $d$ together, independently. From relation (2) we have the following bound

$$sup_{x \in [0,1]} |F_n^\dagger(x;r) - B_n(F_n^\dagger(x;r))| \leq \frac{1}{2n} x(1-x) \|F_n^\dagger\| \leq \frac{1}{8n} \|F_n^\dagger\|, \tag{17}$$

then it is enough to find a bound for $sup_{x_i \in [0,1]} |F^\dagger(x_i;r) - F_n^\dagger(x_i;r)|$. For numerically solving integral equations (7) and (8) by using Bernstein's approximation, because from Theorem 1 we know that for any $F^\dagger \in C[0,1]$ and for any $\epsilon > 0$, there exists $n$ such that the inequality $\|B_n(F^\dagger) - F^\dagger\| < \epsilon$, holds so we can write integral equations (7) and (8) as

$$G^c(x;r) = \lambda \int_0^x k(x,t) B_n(F^c(t;r)) dt, \;\; 0 \leq x \leq 1, \;\; 0 \leq r \leq 1,$$

and

$$G^d(x;r) = \lambda \int_0^x |k(x,t)| B_n(F^d(t;r)) dt, \;\; 0 \leq x \leq 1, \;\; 0 \leq r \leq 1,$$

If we substitute $F_n^\dagger(x;r)$ instead of $F^\dagger(x;r)$ in above equations then the right-hand side of integral equation is exchanged by a new function that we denote it by $\hat{G}^\dagger(x;r)$. So we have,

$$\hat{G}^c(x;r) = \lambda \int_0^x k(x,t) B_n(F_n^c(t;r)) dt, \;\; 0 \leq x \leq 1, \;\; 0 \leq r \leq 1,$$

and

$$\hat{G}^d(x;r) = \lambda \int_0^x |k(x,t)| B_n(F_n^d(t;r)) dt, \;\; 0 \leq x \leq 1, \;\; 0 \leq r \leq 1.$$

Consequently we have

$$sup_{x_i\in[0,1]}|F^c(x_i;r)-F_n^c(x_i;r)| \le \|A_1^{-1}\| max|G^c(x_i;r)-\hat{G}^c(x_i;r)|,\quad i=0,1,...,n, \tag{18}$$

and

$$sup_{x_i\in[0,1]}|F^d(x_i;r)-F_n^d(x_i;r)| \le \|A_2^{-1}\| max|G^d(x_i;r)-\hat{G}^d(x_i;r)|,\quad i=0,1,...,n, \tag{19}$$

where $x_i=\frac{i}{n},\ i=0,1,...,n$. For finding a bound for

$$max|G^\dagger(x_i;r)-\hat{G}^\dagger(x_i;r)|,$$

we let

$$G^c(x;r)=\lambda\int_0^x k(x,t)F^c(t;r))dt,$$
$$G^d(x;r)=\lambda\int_0^x |k(x,t)|F^d(t;r))dt,$$

and

$$\hat{G}^c(x;r)=\lambda\int_0^x k(x,t)B_n(F^c(t;r))dt,$$
$$\hat{G}^d(x;r)=\lambda\int_0^x |k(x,t)|B_n(F^d(t;r))dt.$$

So that

$$\textstyle\int_0^x \lambda k(x,t)(F^c(t;r)-B_n(F^c(t;r)))dt+\int_0^x \lambda k(x,t)B_n(F^c(t;r))dt=G^c(x;r),$$

and

$$\textstyle\int_0^x \lambda|k(x,t)|(F^d(t;r)-B_n(F^d(t;r)))dt+\int_0^x \lambda|k(x,t)|B_n(F^d(t;r))dt=G^d(x;r),$$

then

$$sup_{x\in[0,1]}|G^c(x;r)-\hat{G}^c(x;r)|=sup_{x\in[0,1]}|\textstyle\int_0^x \lambda k(x,t)(F^c(t;r)-B_n(F^c(t;r)))dt|$$
$$\le sup_{x\in[0,1]}(|\lambda k(x,t)||(F^c(t;r)-B_n(F^c(t;r)))|)$$

and

$$sup_{x\in[0,1]}|G^d(x;r)-\hat{G}^d(x;r)|=sup_{x\in[0,1]}|\textstyle\int_0^x \lambda|k(x,t)|(F^d(t;r)-B_n(F^d(t;r)))dt|$$
$$\le sup_{x\in[0,1]}(|\lambda k(x,t)||(F^d(t;r)-B_n(F^d(t;r)))|)$$

if we let $sup_{x,t\in[0,1]}|\lambda k(x,t)|=M$, then we have

$$max_{x_i\in[0,1]}|G^\dagger(x_i;r)-\hat{G}^\dagger(x_i;r)| \le \tfrac{1}{8n}M\|F''^d\|,$$

so by substituting this bound in the inequality (18) and (19) we have,

$$sup_{x_i\in[0,1]}|F^c(x_i;r)-F_n^c(x_i;r)| \le \frac{1}{8n}M\|A_1^{-1}\|\|F''^c\|, \tag{20}$$

and

(21) $$sup_{x_i\in[0,1]}|F^d(x_i;r)-F_n^d(x_i;r)|\le\frac{1}{8n}M\|A_2^{-1}\|\|F''^d\|,$$

then from relations (16), (17),(20) and (21) we have

(22) $$sup_{x_i\in[0,1]}|F^c(x_i;r)-B_n(F_n^c(x_i;r))|\le\tfrac{1}{8n}(M\|A_1^{-1}\|\|F''^c\|+\|F_n''^c\|),$$

and

(23) $$sup_{x_i\in[0,1]}|F^d(x_i;r)-B_n(F_n^d(x_i;r))|\le\tfrac{1}{8n}(M\|A_2^{-1}\|\|F''^d\|+\|F_n''^d\|),$$

therefore by (22), (23) and Remark 1 we have

$$sup_{x_i\in[0,1]}|\underline{F(x_i;r)}-\underline{B_n(F_n(x_i;r))}|\le\tfrac{1}{8n}(M\|A_1^{-1}\|\|F''^c\|+$$

$$\|F_n''^c\|)+\tfrac{1}{8n}(M\|A_2^{-1}\|\|F''^d\|+\|F_n''^d\|),$$

$$sup_{x_i\in[0,1]}|\overline{F(x_i;r)}-\overline{B_n(F_n(x_i;r))}|\le\tfrac{1}{8n}(M\|A_1^{-1}\|\|F''^c\|+$$

$$\|F_n''^c\|)+\tfrac{1}{8n}(M\|A_2^{-1}\|\|F''^d\|+\|F_n''^d\|),$$

hence for all $r\in[0,1]$

$$max\{sup_{x_i\in[0,1]}[|\underline{F(x_i;r)}-\underline{B_n(F_n(x_i;r))}|,|\overline{F(x_i;r)}-\overline{B_n(F_n(x_i;r))}|]\}$$

$$\le\tfrac{1}{8n}(M\|A_1^{-1}\|\|F''^c\|+\|F_n''^c\|)+\tfrac{1}{8n}(M\|A_2^{-1}\|\|F''^d\|+\|F_n''^d\|),$$

and then

$$sup_{x_i\in[0,1]}D(F(x_i),B_n(F_n(x_i)))\le$$

$$sup_{r\in[0,1]}[\tfrac{1}{8n}(M\|A_1^{-1}\|\|F''^c(r)\|+\|F_n''^c(r)\|)+$$

$$\tfrac{1}{8n}(M\|A_2^{-1}\|\|F''^d(r)\|+\|F_n''^d(r)\|)],$$

and the proof is completed. □

## 5. Numerical examples

To illustrate the technique proposed in this paper, consider the following examples.

**Example 5.1.** We consider first kind fuzzy Volterra integral equation with a regular kernel given by,

$$\lambda\int_0^x k(x,t)F(t)dt=G(x),\quad 0\le x\le 1,$$

where $\lambda = 1,\ k(x,t) = \frac{1}{(x^2+t^2)^{\frac{1}{2}}}$ and $G(x) = (\underline{G(x;r)}, \overline{G(x;r)}) = (rx, (2-r)x),\ 0 \leq r \leq 1$. The exact solution in this case is given by $F(x) = (\underline{F(x;r)}, \overline{F(x;r)}) = (r\frac{x}{2^{\frac{1}{2}}-1}, (2-r)\frac{x}{2^{\frac{1}{2}}-1}),\ 0 \leq r \leq 1$. We can see that

$$G^c(x;r) = x,\quad G^d(x;r) = x - rx,\quad 0 \leq r \leq 1.$$

According to Eqs. (7) and (8) we have the following two crisp Volterra integral equations

(24) $$\int_0^x \frac{1}{(x^2+t^2)^{\frac{1}{2}}} F^c(t;r)dt = x,\quad 0 \leq x \leq 1,\quad 0 \leq r \leq 1,$$

(25) $$\int_0^x \frac{1}{(x^2+t^2)^{\frac{1}{2}}} F^d(t;r)dt = x(1-r),\quad 0 \leq x \leq 1,\quad 0 \leq r \leq 1.$$

Now we approximate the unknown functions $F^c(x;r)$ and $F^d(x;r)$ by $B_n(F^c(x;r))$ and $B_n(F^d(x;r))$ for $n = 1, 2, 3$.

We choose $x_0 = 10^{-10}$ and $x_1 = 1 - x_0$. For this example, we use $r = 0, 0.1, \ldots, 1$, where we calculate the error of the exact solution and obtained solution of fuzzy Volterra integral equation with Bernstein approximation. Table 1 show the convergence behavior for $n = 1$. The exact and obtained solution of fuzzy Volterra integral equation in this example at $x = 0.5$ for $n = 1$ is shown in Figure 1.

**Example 5.2.** We consider the fuzzy Abel integral equation with weak singularity given by,

$$\lambda \int_0^x k(x,t)F(t)dt = G(x),\quad 0 \leq x \leq 1$$

where $\lambda = 1$ , $k(x,t) = \frac{1}{\sqrt{x-t}}$ and $G(x) = (\underline{G(x;r)}, \overline{G(x;r)}) = (x^5(r^2+r), x^5(4-r^3-r)),\ 0 \leq r \leq 1$. The exact solution in this case is given by $F(x) = (\underline{F(x;r)}, \overline{F(x;r)}) = ((\frac{1280}{315\pi}x^{\frac{9}{2}})(r^2+r), (\frac{1280}{315\pi}x^{\frac{9}{2}})(4-r^3-r)),\ 0 \leq r \leq 1$. We can see that

$$G^c(x;r) = \frac{x^5(4+r^2-r^3)}{2},\quad G^d(x;r) = \frac{(x^5)(4-r^3-r^2-2r)}{2},\quad 0 \leq r \leq 1.$$

According to Eqs. (7) and (8) we have the following two crisp Volterra integral equations

(26) $$\int_0^x sinh(x)F^c(t;r)dt = \frac{x^5(4+r^2-r^3)}{2},\quad 0 \leq x \leq 1,\quad 0 \leq r \leq 1,$$

(27) $$\int_0^x sinh(x)F^d(t;r)dt = \frac{(x^5)(4-r^3-r^2-2r)}{2},\quad 0 \leq x \leq 1,\quad 0 \leq r \leq 1.$$

Now we approximate the unknown functions $F^c(x;r)$ and $F^d(x;r)$ by $B_n(F^c(x;r))$ and $B_n(F^d(x;r))$ for $n = 1, 2, 3$.

We choose $x_0 = 10^{-10},\ x_j = \frac{j}{n},\ j = 1, ..., n-1$ and $x_n = 1 - x_0$ for $n = 1, 2, 3$. For this example, we use $r = 0, 0.1, \ldots, 1$, where we calculate the error of the exact solution and obtained solution of fuzzy Abel integral equation with Bernstein approximation. Table 1 show the convergence behavior for $n = 1, 2, 3$. The exact and obtained solution of fuzzy Volterra integral equation in this example at $x = 0.5$ for $n = 1, 2, 3$, are shown in Figure 2.

## 6. Summary and conclusions

Here a very simple and straight method, based on approximation of the fuzzy unknown function of an fuzzy Volterra integral equation on the Bernstein polynomial basis is used. Our achieve results in this paper, show that Bernstein's approximation method for solving fuzzy Volterra integral equations of first kind even with singularity, is very effective and the answers are trusty and their accuracy are high and we can execute this method in a computer simply.

## References


[1] S. Abbasbandy, E. Babolian, M. Alavi, Numerical method for solving linear Fredholm fuzzy integral equations of the second kind, Chaos Solitons & Fractals, 31 (2007) 138-146.

[2] E. Babolian, H.S. Goghary, S. Abbasbandy, Numerical solution of linear Fredholm fuzzy integral equations of the second kind by Adomian method, Applied Mathematics and Computation, 161 (2005) 733-744.

[3] C.T.H. Baker, A perspective on the numerical treatment of Volterra equations, J. Comput. Appl. Math., 125 (2000) 217-249.

[4] M.I. Berenguer, D.Gamez, A.I. Garralda-Guillem, M. Ruiz Galan, M.C. Serrano Perez, Biorthogonal systems for solving Volterra integral equation systems of the second kind, J. Comput. Appl. Math., 235 (2011) 1875-1883.

[5] S.S.L. Chang, L. Zadeh, On fuzzy mapping and control, IEEE Trans. System Man Cybernet, 2 (1972) 30-34.

[6] Y. Chen, T. Tang, Spectral methods for weakly singular Volterra integral equations with smooth solutions, J. Comput. Appl. Math., 233 (2009) 938-950.

[7] R.A. Devore, G.G. Lorentz, Constructive approximation, Berlin: Springer-Verlag; 1993.

[8] D. Dubois and H. Prade, Operations on fuzzy numbers, J. Systems Sci. 9 (1978) 613-626.

[9] D. Dubois, H. Prade, Towards fuzzdifferential calculus, Fuzzy Sets Systems, 8 (1982) 1-7.

[10] R. Goetschel, W. Vaxman, Elementary calculus, Fuzzy sets Syst., 18 (1986) 31-43.

[11] A. Kaufmann and M.M. Gupta, Introduction Fuzzy Arithmetic, Van Nostrand Reinhold, New York, 1985.

[12] O. Kaleva, Fuzzy differential equations, Fuzzy Sets Syst. 24 (1987) 301-317.

[13] J.P. Kauthen, Continuous time collocation method for Volterra-Fredholm integral equations, Numer. Math. 56 (1989) 409-424.

[14] G.J. Klir, U.S. Clair, B. Yuan, Fuzzy set theory: foundations and applications. Prentice-Hall Inc.;1997.

[15] P. Linz, Analytical and numerical methods for Volterra equations, SIAM, Philadelphia, PA, 1985.

[16] M. Ma, M. Friedman and A. Kandel, A new fuzzy arithmetic, Fuzzy Sets Syst. 108 (1999) 83-90.

[17] K. Maleknejad, E. Hashemizadeh, R. Ezzati, A new approach to the numerical solution of Volterra integral equations by using Bernstein's approximation, Commun Nonlinear Sci Numer Simulat, 16 (2011) 647-655.

[18] J.Y. Park, H.K. Han, Existence and uniquencess theorem for a solution of fuzzy Volterra integral equations, Fuzzy Sets nad Systems, 105 (1999) 481-488.

[19] M.J.D. Powel, Approximation theory and methods, Cambridge University Press; 1981.

[20] M.L. Puri, D. Ralescu, Fuzzy random variables, J Math. Anal. Appl., 114 (1986) 409-422.

[21] H.H. Sorkun, S. Yalcinbas, Approximate solutions of linear Volterra integral equation systems with variable coefficients, Applied Mathematical Modelling, 34 (2010) 3451-3464.

[22] E. Voronovskaya, Determination de la forme asymptotique dapproximation des functions par les polynomes de M. Bernstein, Doklady Akademii Nauk SSSR, 1932. p. 79-85.

[23] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Inform. Sci. 8 (1975) 199-249.

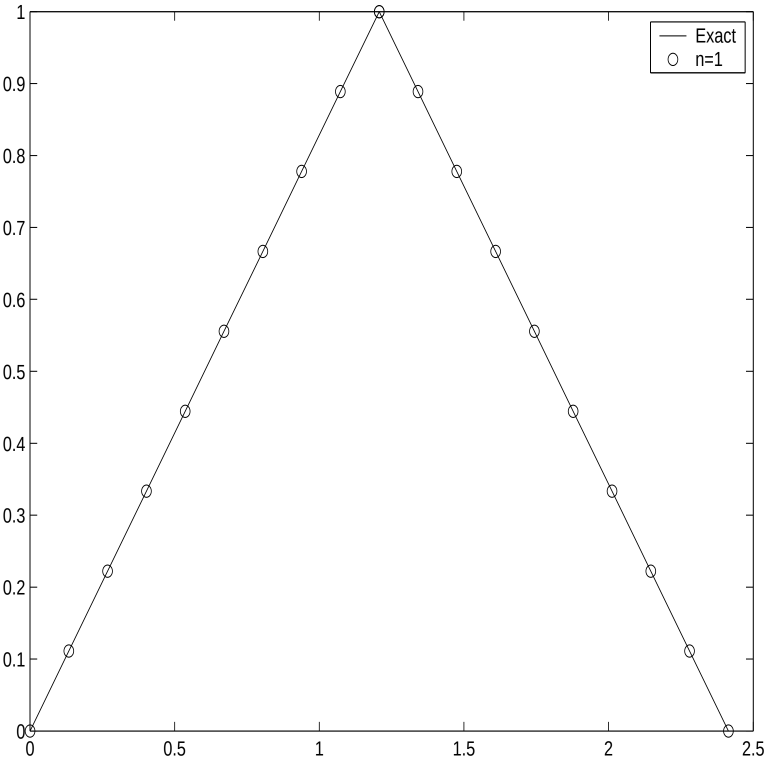


Figure 1. Compares the exact solution and obtained solutions of Bernstein approximation at $x = 0.5$.

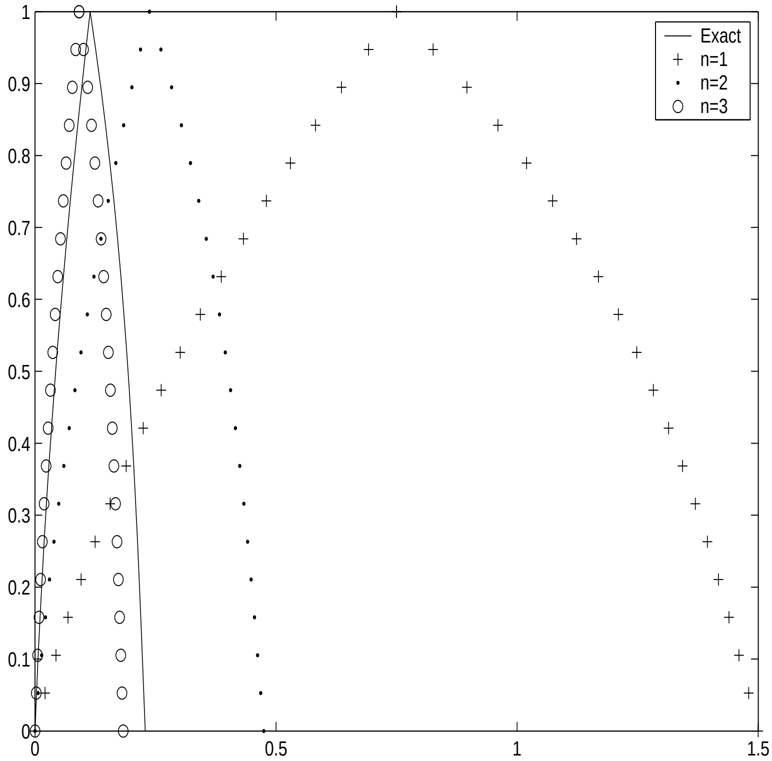


Figure 2. Compares the exact solution and obtained solutions of Bernstein approximation at $x = 0.5$.

| $n$ | Example 5.1 | Example 5.2 |
|---|---|---|
| 1 | 2.0601E-10 | 0.63567 |
| 2 | | 0.12289 |
| 3 | | 0.022849 |

Table 1. Computed error for Examples $5.1 - 5.2$.