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ONE APPROACH TO SOLUTIONS OF MEASUREMENT'S INVERSE PROBLEMS

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ABSTRACT. The investigation of approximate solutions of inverse problems is given in work. For obtaining of the useful information about the exact solution of an inverse problem of measurement a special hypothesis is offered. Two practical inverse problems of measurement are considered where the hypothesis is used: inverse problem of Le Verrier and identification of unbalance characteristics of rotor. For obtaining of stable solutions of these problems a various statements have been considered. Numerical calculation of real problems with application of regularization method is performed.
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1. Introduction

Good coincidence of mathematical results and properties of real physical processes are surprising for many researchers who studied the practical problems with the use of mathematical methods. It is known that the same approaches can correctly describe many physical phenomena which are not clearly related. For example, the Laplace equation is describing well the physical processes of the stationary propagation of heat, the stationary fluid flow and even some of the economic processes [1].

And the amazing thing is that mathematics operates using abstract geometrical objects like of points, lines, planes etc. Such objects do not exist in nature, but solutions of practical problems with the use of these abstract objects correspond to reality quite well.

Such specific quality of mathematics put it in a privileged position among the other sciences.

2. Physical determinism of real processes

To explain this position of mathematics we return to physical determinism of real processes properties [2], [3]. Briefly, this property can be formulated as follows: small changes in the initial data in the traditional problems correspond to small changes in characteristics of the physical process. This fact can be proven by appropriate stability theorems described by mathematical language [1]. The mathematics only fixes these properties of physical processes but does not create them. We will call this property as a property P.

In the set of inverse problems exist some of them in which the error of a mathematical model of a real physical process it is necessary to taken into account [4], [5]. Such type of inverse problems were named as inverse problems of measurement (inverse problems of interpretation or inverse problems of recognition) [6].

Thus, in the process of research of unreal objects that are close to the corresponding characteristics of real objects, based on the property P, can be obtained results that agree well with practice.

Let the mathematical problem of some physical process is represented in the form:

(1) $$\tilde{A}z = u,$$

where $z \in Z, u \in U(Z, U$ are functional spaces), $\tilde{A}$ is approximate operator of process; z is, as a rule, external load, function u is result of external load on process (as response on it).

The property P initiates to appearance specific property of operator $\tilde{A}$. As a rule, this operator is compact operator [2], [7]. In this case the problem of function u determination with initial data $z, \tilde{A}$, will be stable with respect to small changes of these initial data ($propertyP$).

The requirement for stability of the solution of a mathematical problem is indispensable item in the definition of the correctness of the mathematical problem, formulated by J. Hadamard at the beginning of XX-th century [2], [7]. Lack of stability leads to considerable difficulties in approximate calculations.

In this case we have the sets of approximate initial data:

(2) $$z \in Z_{in} = \{z\}, \tilde{A} \in K_A = \{\tilde{A}\},$$

Each pair of source data corresponds to one or more functions u. For the simplicity, we assume that the function u is uniquely determined. Many functions u will form a lot of different responses. Two pairs of initial data $\{z, \tilde{A}\}, z \in Z_{in}, \tilde{A} \in K_A$ which are close to each other $\{z_1, \tilde{A}_1\}, \{z_2, \tilde{A}_2\}$, will give the pair of the close responses u_1, u_2 . Naturally, the concepts of the close responses can be different.

3. Specificity of inverse problems

Let's consider the inverse problem presented in the form (1), where the function z is unknown, and the input is characterized by the couple $\{u, \tilde{A}\}, u \in U_{in}, \tilde{A} \in K_A$.

If the operator $\tilde{A}$ is compact then close pairs of initial data u_1, u_2 can not give close solutions of the inverse problem $z_1, z_2, \tilde{A}_1 z_1 = u_1, \tilde{A}_1 z_2 = u_2$. It can be shown that inverse operator $\tilde{A}^{-1}$ corresponding to compact operator $\tilde{A} \in K_A$ is not continuous [2], [7].

Inverse problems can be divided in two classes: the problem of synthesis and measurement's problems (problems of interpretation) [4], [5], [6]. In case of synthesis problems the obtaining solution $\tilde{z}$ is used later for forecast of function $\tilde{u} = \tilde{A}\tilde{z}$. It is a problem of receiving a function $\tilde{u}$ that will be close to a given one u_{giv} [4], [6]. In the synthesis problems the size of error of the solution obtained is not important. The function $\tilde{u}$ will be close to the desired function u_{giv} due to compactness of operator $\tilde{A}$ (property P). For this reason the approximate solution of inverse problems of synthesis are suitable for further use in spite of their differences among themselves because of the instability.

The final goal in the measuring problems is to obtain a function $\tilde{z}$ that is close to real function of external load z_{ex}. In this case, the error of the solution of the inverse problem with respect to the exact function will be essential [4], [5]. However, it is impossible to accurately describe the physical process with the use of mathematical methods (approximate methods), and to get the exact relationship $u_{ex} = A_{ex} z_{ex}$. Therefore, it is impossible, in principle, to construct an approximate solution of the inverse problem of measurement $\tilde{z}$ which would be close to absolutely exact solution of the inverse problem z_{ex}. Furthermore, it is impossible to estimate the error of approximate solution $\tilde{z}$ with respect to the absolutely exact solution z_{ex}.

The algorithm for constructing an approximate solution based on the Tikhonov regularization method is proposed in works [2], [7], [8], [9].

Let the functions z, u belong to a Banach functional spaces $z \in Z, u \in U$. Further assume that the exact operator A_{ex} is linear. Suppose also that deviations of initial data $\{\tilde{u}, \tilde{A}\}$ with respect the exact data $\{u_{ex}, A_{ex}\}$ are given:

$$\| \tilde{A} - A_{ex} \|_{Z \to U} \leq h, \| \tilde{u} - u_{ex} \|_U \leq \delta. \tag{3}$$

Solution of the inverse problem is reduced to solution of following extreme problem:

(4) $$\inf_{z\in Z_1} M^{\alpha}[z, u_\delta, \tilde{A}] = \inf_{z\in Z_1} \| \tilde{A}z - u_\delta \|_U^2 + \alpha\Omega[z] = M^{\alpha}[z_\alpha, u_\delta, \tilde{A}],$$

where $\Omega[z]$ is stabilizing functional for equation [2], [7]. Regularization parameter α can be determined by the method of generalized discrepancy:

(5) $$\| \tilde{A}z_\alpha - u_\delta \|_U^2 = (\delta + h\|z_\alpha\|)^2 + \mu^2(\tilde{u}, \tilde{A}),$$

where $\mu(\tilde{u}, \tilde{A})$ is a degree of inconsistency.

However, there's no way to determine the error h because the exact operator A_{ex} is unknown. Furthermore, the exact operator A_{ex} can not be built in principle as far as mathematical methods only approximately describe the real processes.

Of course, with certain assumptions regarding of exact operator A_{ex} it is possible to get some estimate of error, however, such an assessment will be unreal.

Therefore, the approximate solution of inverse problems of measurement are not of interest for practical use because of the instability of such solutions.

Way out of this impasse exists if we instead of solution of inverse problem will limited only to obtaining the estimates of the exact solution.

4. Main Hypothesis

To obtain useful information about the exact solution of the inverse problem of measurement is suggested the following hypothesis [10], [11], [12]: for absolutely exact solution of the inverse problem, the inequality for any operator in the approximate equation of the inverse problem (1) is valid and has the form

(6) $$\Omega[z_{ex}] \geq \Omega[z_\alpha]$$

where z_{ex} there is an exact solution of the inverse problem of measurement, z_α is the regularized solution of the inverse problem with the operator $\tilde{A}$. It is supposed that operator $\tilde{A}$ in (1) describes adequately process. If the exact operator is linear, then the inequality (6) is obvious.

Basic research efforts of inverse problems investigation are transferred to the sphere of evidence for the existence, uniqueness and stability estimates of exact solutions of measurement's inverse problems [10], [11], [12].

Theorem 1. *If the functional space Z is reflex Banach space, the functional $\Omega[z_\alpha]$ is convex and lower semi-continuous on Z, Lebesgue's set for some function from $Q_{p,\delta} \in Z$ is bounded then the function $z_\alpha \in Q_{p,\delta}$ exists.*

To illustrate the use of this hypothesis we consider two practical problems of measurement: inverse problem of astrodynamics and the problem of identification of technological resistance on the rolling mill of sheets [11], [13], [14].

5. Inverse problem of astrodynamics

The problem of determining the position of an unknown gravitational mass with the use of results of the processing of the observed perturbations in the motion of celestial bodies, caused by this mass, is investigated.

At first, this problem was considered by J. D. Adams and Le Verrier in 1845-1846. In this paper the another approach was suggested which provides the greater universatility.

Let us consider n interacting masses moving under the forces of mutual attraction in an inertial coordinate system. Denote masses m_i the indexes $i(i = 1, 2, 2, .., n), \vec{r}_{ik}$ denote the vectors connecting the mass m_i with mass m_k. According to Newton's law on mass m_j acts the resultant force equal

$$\vec{F}_j = G \sum_{i=1, i \neq j}^{n} \frac{m_i m_j}{|\vec{r}_{ij}|^3} \vec{r}_{ij}, \tag{7}$$

where G is the gravitational constant.

Under this force the motion of mass m_j is described by differential equation

$$\frac{d^2 \vec{r}_{0j}(t)}{dt^2} = \frac{1}{m_j} \vec{F}_j, \tag{8}$$

where $\vec{r}_{0j}$ is the radius vector joining the origin of the inertial coordinate system with mass m_j.

Let us transform the right part of equation (8) to variables $\vec{r}_{0j}, j = 1, 2, 3, ., n$.

$$\frac{d^2 \vec{r}_{0j}(t)}{G dt^2} = \frac{1}{G m_j} \vec{F}_j = \sum_{i=1, i \neq j}^{n} \frac{m_i}{|\vec{r}_{oi} - \vec{r}_{oj}|^3} (\vec{r}_{0i} - \vec{r}_{0j}), \tag{9}$$

It is assumed that among n gravitational masses only the location of mass m_n is unknown. Then in (9) the last term in the sum on the right part is uncertain.

Equation (9) takes the form

$$\frac{d^2\vec{r}_{0j}(t)}{Gdt^2} = \sum_{i=1,i\neq j}^{n-1} \frac{m_i}{|\vec{r}_{oi}-\vec{r}_{oj}|^3}(\vec{r}_{0i}-\vec{r}_{0j}) - \bar{f}_j(t), j \neq n, \tag{10}$$

where

$$\bar{f}_j(t) = \frac{m_n}{|\vec{r}_{on}-\vec{r}_{oj}|^3}(\vec{r}_{0n}-\vec{r}_{0j})$$

is the unknown function.

In projections on the inertial coordinate system equations (10) can are written as:

$$\frac{d^2x_{0j}(t)}{Gdt^2} = \sum_{i=1,i\neq j}^{n-1} \frac{m_i}{|\vec{r}_{oi}-\vec{r}_{oj}|^3}(\vec{r}_{0i}-\vec{r}_{0j})_x - \vec{f}_{jx}(t), j \neq n, \tag{11}$$

$$\frac{d^2y_{0j}(t)}{Gdt^2} = \sum_{i=1,i\neq j}^{n-1} \frac{m_i}{|\vec{r}_{oi}-\vec{r}_{oj}|^3}(\vec{r}_{0i}-\vec{r}_{0j})_y - \vec{f}_{jy}(t), j \neq n, \tag{12}$$

$$\frac{d^2z_{0j}(t)}{Gdt^2} = \sum_{i=1,i\neq j}^{n-1} \frac{m_i}{|\vec{r}_{oi}-\vec{r}_{oj}|^3}(\vec{r}_{0i}-\vec{r}_{0j})_z - \vec{f}_{jz}(t), j \neq n, \tag{13}$$

where $\vec{r}_{0j} = \vec{i}x_{0j} + \vec{j}y_{0j} + \vec{k}z_{0j}$; $\vec{f}_{jx}, \vec{f}_{jy}, \vec{f}_{jz}$ - the projections of $\vec{f}_j$ on the corresponding axis of inertial coordinate system,$(\vec{r}_{0i}-\vec{r}_{0j})_x, (\vec{r}_{0i}-\vec{r}_{0j})_y, (\vec{r}_{0i}-\vec{r}_{0j})_z$ - the similar projections of the vector $(\vec{r}_{0i}-\vec{r}_{0j})$.

Integrate equation (11), (12), (13) twice from t_0 to t we obtain:

$$\begin{aligned}
\frac{x_{0j}(t)}{G} &= \int_{t_0}^{t} \mu_1(\tau)(t-\tau)d\tau + \int_{t_0}^{t} f_{jx}(\tau)(t-\tau)d\tau + \frac{\dot{x}_{0j}(t_0)}{G} + \frac{x_{0j}(t_0)}{G}, \\
\frac{y_{0j}(t)}{G} &= \int_{t_0}^{t} \mu_2(\tau)(t-\tau)d\tau + \int_{t_0}^{t} f_{jy}(\tau)(t-\tau)d\tau + \frac{\dot{y}_{0j}(t_0)}{G} + \frac{y_{0j}(t_0)}{G}, \\
\frac{z_{0j}(t)}{G} &= \int_{t_0}^{t} \mu_3(\tau)(t-\tau)d\tau + \int_{t_0}^{t} f_{jz}(\tau)(t-\tau)d\tau + \frac{\dot{z}_{0j}(t_0)}{G} + \frac{z_{0j}(t_0)}{G},
\end{aligned} \tag{14}$$

where

$$\mu_1(t) = \sum_{i=1,i\neq j}^{n-1} \frac{m_i}{|\vec{r}_{oi}-\vec{r}_{oj}|^3}(\vec{r}_{0i}-\vec{r}_{0j})_x, \mu_2(t) = \sum_{i=1,i\neq j}^{n-1} \frac{m_i}{|\vec{r}_{oi}-\vec{r}_{oj}|^3}(\vec{r}_{0i}-\vec{r}_{0j})_y,$$

$$\mu_3(t) = \sum_{i=1,i\neq j}^{n-1} \frac{m_i}{|\vec{r}_{oi}-\vec{r}_{oj}|^3}(\vec{r}_{0i}-\vec{r}_{0j})_z$$

.

Let us represent each equation of system (14) in the form

$$\int_{t_0}^{t}(t-\tau)f_{jk}(\tau)d\tau = u_{jk}(t), k = 1, 2, 3, \tag{15}$$

where

$$u_{j1}(t) = \frac{x_{0j}(t)}{G} - \int_{t_0}^{t}\mu_1(\tau)(t-\tau)d\tau - \frac{\dot{x}_{0j}(t_0)}{G} - \frac{x_{0j}(t_0)}{G},$$

$$u_{j2}(t) = \frac{y_{0j}(t)}{G} - \int_{t_0}^{t}\mu_2(\tau)(t-\tau)d\tau - \frac{\dot{y}_{0j}(t_0)}{G} - \frac{y_{0j}(t_0)}{G},$$

$$u_{j3}(t) = \frac{z_{0j}(t)}{G} - \int_{t_0}^{t}\mu_3(\tau)(t-\tau)d\tau - \frac{\dot{z}_{0j}(t_0)}{G} - \frac{z_{0j}(t_0)}{G}.$$

Equations (15) are Voltera integral equations of the first kind with respect to the unknown functions $f_{jk}(t), k = 1, 2, 3$.

We can restore the vector force $\vec{f_j}(t)$ exerted by the mass m_n of mass m_j up to a constant factor if the solutions of equations (15) $f_{jk}(t), k = 1, 2, 3$ were obtained.

We obtain the force $\vec{f_l}(t)$ acting on the mass m_l from the mass m_n if we solved similar equations of the type (15) for the mass with the number l determined (up to a constant factor). The intersection of the lines of action $\vec{f_j}(t)$ and $\vec{f_l}(t)$ gives the position of mass m_n in space (in the chosen inertial system).

As is easily seen that the functions $u_{jk}(t)$ are defined from astronomical observations $\vec{r}_{0l}(t), i = 1, 2, , ..(n-1)$ of the motions of the masses $m_j(t), j = 1, 2, , ..(n-1)$ with some error.

Right parts of equation (15) are continuous functions which belong to a normed space $C[t_0, T]$, where $[t_0, T]$ is a period of time in which the movement of mass m_n is investigated.

Solutions of equation (15) in the physical sense must also belong to $C[t_0, T]$, that is $f_{jk}(t) \in C[t_0, T]$. Under these conditions, the solution of equations (15) is ill-posed problem [2].

In the equations of motion (7) coefficients $m_i(i = 1, 2, .., (n-1)), G$ are determined from astronomical observations and experimental studies, and so these values are given approximate. It assumes that each coefficient in equations (7) can take values in some interval:

$$0 < m_i^0 \leq m_i \leq m_i^{up}, i = \overline{1,(n-1)}, i \neq j, 0 < G^0 \leq G \leq G^{up}. \tag{16}$$

Let us introduce the following notations

$$\vec{p} = (b_1, b_2, .., b_{n-1})^*, \vec{R}(t) = (r_{01}(t), r_{02}(t), , .., r_{0(n-1)}(t),)^*,$$

where $b_1 = m_1, ..., b_{j-1} = m_{j-1}, b_j = m_{j+1}, ..., b_{n-2} = m_{n-1}, b_{n-1} = \frac{1}{G}$; $(.)^*$ is the sign of transposition.

Inequalities (16) define a closed area $\bar{D}$ in $(n-1)$ - dimensional Evclide's space R^{n-1}. Set of vector functions $\vec{R}(t)$ form a linear function space $C_n[t_0, T]$, which can introduce the norm as follows [2]:

$$\| \vec{R}_1 - \vec{R}_2 \|_{C_n[t_0,T]} = \max_{1 \le i \le n} |r_{1,i} - r_{2,i}|,$$

where

$$r_i^k = \| r_i^k \|_{C[t_0,T]} = \max_{t \in C[t_0,T]} |r_{0i}^k(t)|, \vec{R}_k(t) = (r_{01}^k, r_{02}^k, .., r_{0n}^k)^*, k = 1, 2.$$

Let us transform the equation (15) to the form

$$\tilde{A}f = u = B_p \vec{R}, \tag{17}$$

where B_p is a bounded linear operator carrying elements of the functional space $C_n[t_0, T]$ into $C[t_0, T]$.

The operator B_p depends on the specific values of the parameters of the mathematical model of the process, i.e. from p , so assume that the operator B_p is given approximate.

We denote $\vec{R}_{ex}(t), u_{ex}$ respectively the exact vector function $\vec{R}(t)$, and the exact function u of the right-hand side of equation (17).

It is suppose that instead $\vec{R}_{ex}(t)$ in (17) is given an approximate initial datum $\vec{R}_\delta(t) = (\tilde{r}_{o1}(t), \tilde{r}_{o2}(t), , .., \tilde{r}_{on}(t),)^*$ for which the inequality

$$\| \vec{R}_{ex}(t) - \vec{R}_\delta(t) \|_{C_n[t_0,T]} \le \delta$$

is valid.

These data $\vec{R}_\delta(t)$ will correspond the approximate value $\tilde{u}$ of the right-hand part of equation (17) $(\tilde{u} = B_p \vec{R}_\delta)$.

We estimate the deviation function $\tilde{u}(t)$ from $u_{ex}(t)$:

$$\| \tilde{u}(t) - u_{ex}(t) \|_{C[t_0,T]} = \| B_p \vec{R}_\delta - B_{ex} \vec{R}_{ex} \|_{C[t_0,T]} \le b_0 \delta + d_1 \| \vec{R}_\delta \| = \delta_0, \tag{18}$$

where

$$b_0 = \sup_{\vec{p} \in \bar{D}} \| \vec{R}_p \|; d_1 \ge \sup_{\vec{p} \in \bar{D}} \| B_{\vec{p}} - B_{ex} \|;$$

B_{ex} is exact operator in (3).

As the real process possible to describe only approximately by mathematical methods, we will also assume that the exact operator A_{ex} in equation (1) (if A_{ex} is linear) differs from of the approximate operator $\tilde{A}$ on value

$$h \ge \| \tilde{A} - A_{ex} \|_{Z \to U} .$$

In this case it is possible to use an algorithm for solving inverse problems with approximate operator $\tilde{A}$ [2], [8], [9].

However, the assumption of linearity exact operators A_{ex}, B_{ex} as well as the information about the values h, d_1 are not justified in many cases.

Consider the set of possible solutions of equation (17) with fixed operators $\tilde{A}, B_p, p \in \bar{D}$:

(19) $$Q_{\vec{p},\delta} = \{f : \| \tilde{A}f - B_{\vec{p}}\vec{R}_\delta \|_{C[t_0,T]} \leq \| B_{\vec{p}} \| \delta\}.$$

Suppose that $\tilde{f}$ is a solution of following extreme problem

(20) $$\Omega[\tilde{f}] = \inf_{f \in Q_{\vec{p},\delta} \bigcap Z_1} \Omega[f].$$

According main hypothesis the inequality for function $\tilde{f}$ is valid

(21) $$\Omega[\tilde{f}] \leq \Omega[f_{ex}], A_{ex}f_{ex} = B_{ex}\vec{R}_{ex}.$$

If $\tilde{f} \neq 0$ then we can say with confidence that there is a real celestial body m_n. It is clear that the function $\tilde{f}$ may differ significantly from the exact solution f_{ex}.

If $\tilde{f} \equiv 0$ then we can say nonetheless that possible exists a real celestial body m_n as operator $B_{\vec{p}}$ is fixed. It is possible that exists such operator $B_{\vec{p}_1}$ among set of operators for which the function $\tilde{f} \neq 0$.

We consider now the union of sets:

(22) $$Q^{un} = \bigcup_{\vec{p} \in \bar{D}} Q_{\vec{p},\delta}.$$

Tikhonov regularization method [2] with a stabilizing functional

(23) $$\Omega[f] = \|f\|^2_{W_2^1[t_0,T]} = \int_{t_0}^{T} [q_0 f^2 + q_1 \dot{f}^2] d\tau, q_0 \geq 0, q_1 > 0$$

is used to solve the equation [2].

Since $Q^{un} \subset Q_{\delta_0}$, then using of more narrow set Q^{un} of possible solutions of equation (17) instead of set Q_{δ_0} will provide a more informative solution. The proposed approach is a continuation of works [15], [16], [17].

We denote f^{un} as the solution of following problem:

(24) $$\Omega[f^{un}] = \inf_{f \in Q^{un}} \Omega[f].$$

To implement such an approach must be able to distinguish among the operators B_p of an operator B_{p_0} such that if the condition

(25) $$\Omega[f_1] = \Omega[\tilde{A}^{-1}B_p\vec{R}], \Omega[f_2] = \Omega[\tilde{A}^{-1}B_{p_0}\vec{R}],$$

then the inequality

$$\Omega[f_1] = \|f_1\|^2_{W_2^1[t_0,T]} \geq \Omega[f_2] = \|f_2\|^2_{W_2^1[t_0,T]} \tag{26}$$

is valid for any possible $\vec{R}$ and anyone $p \in \bar{D}$; $\tilde{A}^{-1}$ is an inverse operator to $\tilde{A}$.

In [15], [16] approximate operator B_{p_0} on the left side of equation (17) with the "opposite" properties named "least rude". Therefore, the operator B_{p_0} will be called "the most rude".

If the operator B_{p_0} exists and is uniquely determined, then the problem of finding the greatest lower bound of the functional $\Omega[f]$ on the set Q^{un} will have a solution that coincides with the solution of the simpler problem [15], [17]: find the element $f_0 \in Q_{p_0,\delta}$ for which the equality

$$\Omega[f^{un}] = \inf_{f \in Q_{p_0,\delta}} \Omega[f]. \tag{27}$$

is valid.

The problem (27) has a solution for any $p_0 \in \bar{D}$ and δ as shown in [2].

Theorem 2. *Crudest operator B_{p_0} in equation (17) exists, is uniquely determined and corresponds to the vector*

$$p_0 = (m_1^0, m_2^0, ..., m_{j-1}^0, m_{j+1}^0, ..., m_{n-1}^0, 1/G^{up})^*.$$

Proof. Let $\vec{R}(t)$ be a realization of astronomical observations. Consider the problem of determining the exact lower bound functional $\Omega[f] = \Omega[\tilde{A}^{-1}B_p\vec{R}]$ in area $\bar{D}$ a fixed $\vec{R}(t)$. Extreme of continuous functional $\Omega[f]$ attained on a vector $p_0 \in \bar{D}$ by the Weierstrass theorem.

When any $p \in \bar{D}$ function $\Omega[\tilde{A}^{-1}B_p\vec{R}]$ is strictly positive since $\Omega[f] = \|f\|^2_{W_2^1[t_0,T]} > 0$ by $f \neq 0, \forall p \in \bar{D}$.

Function $\Omega[f]$ for fixed $\vec{R}(t)$ can be represented as a quadratic form

$$\Omega[f] = (Cp, p) = \Omega[p],$$

where C is a real symmetric matrix $C = (c_{ik})_{k,i=1}^n$.

Matrix coefficients C are given by:

$$c_{ik} = \int_{t_0}^{T} (q_0 a_{ij}(t) a_{kj}(t) + q_1 \frac{da_{ij}}{dt}\frac{da_{kj}}{dt}), i, k = 1, 2, ..(n-1).$$

Since $\Omega[f] > 0$ for any $p \in \bar{D}$ the inequalities Silvester are valid:

$$c_{11} > 0, \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} > 0, \begin{vmatrix} c_{11} & . & . & . & c_{1,n-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ c_{n-1,1} & . & . & .c_{n-1,n-1} & \end{vmatrix} > 0.$$

Necessary and sufficient conditions for strong convexity of $\Omega[\vec{p}]$ on $\bar{D}$ is the following [19]:

(28) $$\sum_{i,k=1}^{(n-1)} \frac{\partial^2 \Omega[p]}{\partial b_i \partial b_k} \xi_i \xi_k > 0,$$

for any $\xi = (\xi_1, \xi_2, ..., \xi_{n-1})^* \in E^{n-1}$ and any $\vec{p} \in \bar{D}$.

Quadratic form (Cp, p) is positive as

$$\bar{C} = \left(\frac{\partial^2 \Omega[p]}{\partial b_i \partial b_k} \xi_i \xi_k \right)_{i,k=1}^{(n-1)} = (\bar{c}_{ik})_{i,k=1}^{(n-1)} = (2c_{ik})_{i,k=1}^{(n-1)}.$$

Therefore $\Omega[\vec{p}]$ is strongly convex on $\bar{D}$.

As in [2] that $\Omega[\vec{p}]$ achieves the greatest lower bound at a single point

$$p_0 = (m_1^0, m_2^0, ..., m_{j-1}^0, m_{j+1}^0, ..., m_{n-1}^0, G^{up-1})^* \in \bar{D}$$

by any $\vec{R}(t)$. The theorem is proved.

If $\tilde{f} \equiv 0$ then we can say nonetheless that possible exists a real celestial body m_n as operator $B_{\vec{p}}$ is fixed. It is possible that exists such operator $B_{\vec{p}_1}$ among set of operators for which the function $\tilde{f} \neq 0$.

Suppose that among the operators B_p we can select some operator B_{p^1} for which conditions is valid

(29) $$\Omega[f_1] = \Omega[\tilde{A}^{-1} B_p \vec{R}], \Omega[f_2] = \Omega[\tilde{A}^{-1} B_{p^1} \vec{R}],$$

(30) $$\Omega[f_1] = \|f_1\|^2_{W_2^1[t_0,T]} \leq \Omega[f_2] = \|f_2\|^2_{W_2^1[t_0,T]}$$

is valid for any $\vec{R}$ and any $p \in \bar{D}$; $\tilde{A}^{-1}$ is inverse operator to $\tilde{A}$.

Operator B_{p^1} in equation (17) we will be name as "special maximal operator" in sense of execution of inequality (30).

If the operator B_{p_1} exists and is uniquely determined, then the problem of finding the greatest lower bound of the functional $\Omega[f]$ on the set Q^{un} will have a solution that coincides with the solution of the simpler problem [17]: find the element $f^1 \in Q_{p^1,\delta}$ for which the equality

$$\Omega[f^{un}] = \sup_{\vec{p}\in\bar{D}} \inf_{f\in Q_{p_0,\delta}\bigcap Z_1} \Omega[f]. \tag{31}$$

is valid.

The problem (31) has a solution for any $p^1 \in \bar{D}$ and δ as the conditions of Theorem 1. are executed.

It is evident that following inequality is valid

$$\Omega[f_T] \geq \Omega[f_1] \geq \Omega[f_{\alpha,p}] \geq \Omega[f_0] \geq \Omega[f^{un}]. \tag{32}$$

Theorem 3. *Special operator B_{p^1} in equation (17) exists, is uniquely determined and corresponds to the vector* $p^1 = (m_1^{up}, m_2^{up}, \dots, m_{j-1}^{up}, m_{j+1}^{up}, \dots, m_{n-1}^{up}, 1/G^0)^*$.

The proof of Theorem 3. is executing similar as Theorem 2.

The estimate obtained leads to the conclusion about the existence with guarantee unknown celestial body (if $f^1 \neq 0$) and the conclusion of his absence (if $f^1 \neq 0$) but with no guarantee. In the second case, the existence of a celestial body is also possible if the refinement of the structure of operators $\tilde{A}, B_{p^1}$ will made.

Other variants are possible estimates of exact solutions.

6. Identification of moment of technological resistance on rolling mill

As the second of measurement's inverse problems the problem of definition of the moment of technological resistance on the rolling mill is considered [11], [20].

The important characteristic of process of rolled metal is the moment of technological resistance on the working barrels of the rolling mill.

The curve of change of technological resistance moment which was obtained by help of plasticity theory was shown on Figure 1. as dotted line.

In paper the problem of definition of the technological resistance moments by a method of identification is considered [11], [20], i.e. method of indirect measurements: on basis of results of measurement of fluctuations of the moments in the main mechanical line of the rolling mill it is necessary to determine the real character of change of the moments of technological resistance. In this case it is necessary to take into account an error of the mathematical description of process of fluctuations.

Mathematical model of motion of the main mechanical line of the rolling mill of sheets was chosen in form:

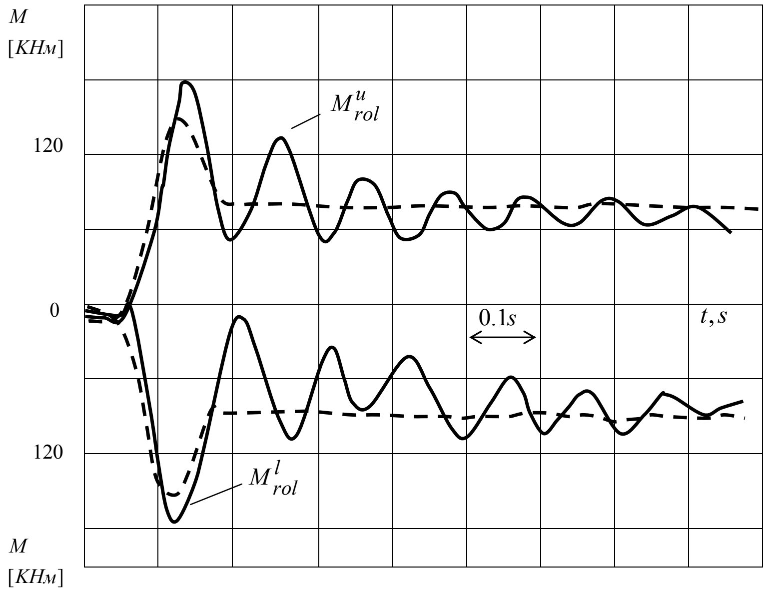

FIGURE 1. Graphics of moments which are acting to worked barrels of list rolling mill at experiment.

$$\ddot{M}_{12} + a_{12}M_{12} + a_{13}M_{23} + a_{14}M_{24} = b_1 J_u,$$

(33) $$\ddot{M}_{23} + a_{23}M_{23} + a_{22}M_{12} + a_{24}M_{24} = b_2 M^u_{rol},$$

$$\ddot{M}_{24} + a_{34}M_{24} + a_{32}M_{13} + a_{33}M_{23} = b_3 M^l_{rol}.$$

where functions $J_u(t), M_{23}(t), M_{24}(t)$ were obtained by experiment [11]. The integral equations such as (17) with the inexact operators are received for definition of unknown functions M^u_{rol}, M^l_{rol}.

The regularization method is used for inverse problem solution of technological resistance moment identification [2]. The initial problem was replaced with the solution of following extreme problem [20]:

(34) $$\Omega[z_*] = \inf_{z \in Q_\delta} \Omega[z],$$

where $Q_\delta = z : z \in Z_1, \|\tilde{A}z - \tilde{u}\|_U \leq \delta, \|u_{ex} - \tilde{u}\|_U \leq \delta, \delta$ is inaccuracy of initial data, u_{ex} is exact initial data.

The functional of kind

(35) $$\Omega[z] = \int_0^T \dot{z}^2 dt$$

was chosen as stabilizing functional.

The solutions of equation (17) with the exact operators were obtained by regularization method. Functions M^u_{rol}, M^l_{rol} are shown on Fig.1 (continuous lines).

On basic of main hypothesis we can take the conclusion that exact solutions have more oscillating characters of solutions as for regularized solutions the following inequalities are valid:

$$\Omega[M^u_{rol,ex}] = \int_0^T \left(\dot{M}^u_{rol,\,ex}\right)^2 dt \geq \Omega[M^u_{rol}] = \int_0^T \left(\dot{M}^u_{rol}\right)^2 dt, \tag{36}$$

$$\Omega[M^l_{rol,ex}] = \int_0^T \left(\dot{M}^l_{rol,ex}\right)^2 dt \geq \Omega[M^l_{rol}] = \int_0^T \left(\dot{M}^l_{rol}\right)^2 dt, \tag{37}$$

where $M^u_{rol,ex}, M^l_{rol,ex}$ are real moments of technological resistance on working barrels of rolling mill of sheets.

Thus as a result of an estimations of the exact solutions the useful information concerning the real moments of technological resistance was received which cannot have character of change as shown on Fig.1 (dotted line).

7. Conclusion.

One of the possible approach for solving inverse problems of measurement when there is no exact initial information is considered. The main hypothesis for estimation of exact solution of measurement inverse problems is suggested. The conditions of existence of estimations were obtained. As the examples two practical problems were investigated: inverse problem of astrodynamics and identification of external moment loads to working barrels of rolling mill of sheets.

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