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SOLVING LINEAR AND NONLINEAR ABEL FUZZY INTEGRAL EQUATIONS BY FUZZY LAPLACE TRANSFORMS

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Abstract. In this paper, we present an efficient analytic method for solving Abel fuzzy integral equation based fuzzy Laplace transforms, so all calculations can be easily implemented. The proposed method in details is discussed and illustrated by solving some examples.

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1. Introduction

Fuzzy integral equations are important for studying and solving a large proportion of the problems in many topics in applied mathematics, in particular in relation to physics, geographic, medical, biology, etc. Usually in many applications some of the parameters in our problems are represented by fuzzy number rather than crisp, and hence it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy integral equations and solve them.

The concept of integration of fuzzy functions was first introduced by Dubois and Prade [6]. Alternative approaches were later suggested by Goetschel and Voxman [13], Kaleva [12], Nanda [14] and others. While Goetschel and Voxman [13] preferred a Rimann integral type approach, Kalva [12] chose to define the integral of fuzzy function, using the Lebesgue type concept for integration. One of the first applications of fuzzy integration was given by Wu and Ma [18] who investigated the Fuzzy Fredholm integral equation of second kind (FFIE-2). This work which established the existence of a unique solution to (FFIE-2) was followed by other work on (FFIE-2) [8] where a fuzzy integral equation replaced an original fuzzy differential equation.

The fuzzy Laplace transform method is practically the most important operational method. The fuzzy Laplace transform method is a powerful technique that can be used

for solving initial value problems and integral equations as well [9, 15 , 16]. Abel integral equation occurs in many branches of scientific fields [17], such as microscopy, seismology, radio astronomy, electron emission, atomic scattering, radar ranging, plasma diagnostics, X-ray radiography, and optical fiber evaluation. In this paper, we introduced Abel fuzzy integral equations. Linear and nonlinear Abel fuzzy integral equations are transformed in such a manner that the fuzzy Laplace transform method can be applied.
The structure of paper is organized as follows: In Section 2, some basic definitions and results which will be used later are brought. In Section 3, we will review fuzzy Laplace transform. In Section 4, we introduce Abel fuzzy integral equations. In section 5, we apply Laplace transform method for solving Abel fuzzy integral equations, then the proposed method is implemented for solving three illustrative examples in Section 6 and finally, conclusion is drawn in Section 7.

2. Preliminaries

We now recall some definitions needed through the paper.

Definition 1. A fuzzy number is a fuzzy set $u : R^1 \to [0, 1]$ which satisfies following conditions

a: u is upper semicontinuous.

b: $u(x) = 0$ outside some interval $[c, d]$.

c: There are real numbers a and b, $c \leq a \leq b \leq d$, for which

i) $u(x)$ is monotonically increasing on $[c, a]$,

ii) $u(x)$ is monotonically decreasing on $[b, d]$,

iii) $u(x) = 1$ for $a \leq x \leq b$.

The set of all fuzzy numbers, as given by definition (1) is denoted by E^1. An alternative definition or parametric form of a fuzzy number which yields the same E^1 is given by Kaleva [12].

Definition 2. A fuzzy number u is a pair $(\underline{u}(r), \overline{u}(r))$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \leq r \leq 1$, satisfying the following requirements:

a: $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,

b: $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,

c: $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $k > 0$, we define addition $(u + v)$ and multiplication by k as:

(1) $$(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r),$$
$$(\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r),$$

(2) $$(\underline{ku})(r) = k\underline{u}(r),$$
$$(\overline{ku})(r) = k\overline{u}(r).$$

The collection of all the fuzzy numbers with addition and multiplication as defined by Eqs. (1) and (2) is denoted by E^1 and is u convex cone. it can be shown that Eqs. (1) and (2) are equivalent to the addition and multiplication as defined by using the $\alpha - cut$ approach [8] and the extension principles [14]. We will next define the fuzzy function notation and a metric D in E^1 [13].

Definition 3. For arbitrary numbers $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$

$$D(u,v) = \max\{\sup_{0\le r\le 1} |\overline{u}(r) - \overline{v}(r)|, \sup_{0\le r\le 1} |\underline{u}(r) - \underline{v}(r)|\},$$

is the distance between u and v [13].

Definition 4. Suppose $f : [a,b] \to E^1$ for each partition $p = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ and for arbitrary ε_i; $x_{i-1} \le \varepsilon_i \le x_i$, $1 \le i \le n$, take

$$\lambda = \max_{1\le i\le n} |x_i - x_{i-1}| \ ,$$

and $R_p = \sum_{i=1}^{n} f(\varepsilon_i)(x_i - x_{i-1})$. The definition integral of $f(x)$ over $[a,b]$ is

$$\int_a^b f(x)dx = \lim_{\lambda\to 0} R_p \ ,$$

provided that this $limit$ exists in the metric D.

If the fuzzy function $f(x)$ is continuous in the metric D, the definite integral exists [13]. Furthermore,

(3) $$(\underline{\int_a^b f(x,r)dx}) = \int_a^b \underline{f}(x,r)dx, \quad (\overline{\int_a^b f(x,r)dx}) = \int_a^b \overline{f}(x,r)dx,$$

where $(\underline{f}(x,r), \overline{f}(x,r))$ is the parametric form of $f(x)$. It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [12]. However, if $f(x)$ is continuous, both approaches yield the same value. Moreover, the representation of the fuzzy integral using Eq. (2) is more convenient for numerical calculations. More details about the properties of the fuzzy integral are given in [12, 13].

Lemma 1 [7]. If f and $g : [a,b] \subseteq R \to E^1$ are fuzzy continuous function, then the function $F : [a,b] \to R_+$ by $F(x) = D(f(x), g(x))$ is continuous on $[a,b]$, and

$$D\left(\int_a^b f(x)dx, \int_a^b g(x)dx\right) \leq \int_a^b D(f(x), g(x))dx. \tag{4}$$

Theorem 1 [18]. Let $f(x)$ be a fuzzy value function on $[a,\infty)$ and it is represented by $(\underline{f}(x,r), \overline{f}(x,r))$. For any fixed $r \in [0,1]$, assume $\underline{f}(x,r)$ and $\overline{f}(x,r)$ are Riemann-integrable on $[a,b]$ for every $b \geqslant a$ and assume there are two positive $\underline{M}(r)$ and $\overline{M}(r)$ such that $\int_a^b |\underline{f}(x,r)|dx \leqslant \underline{M}(r)$ and $\int_a^b |\overline{f}(x,r)|dx \leqslant \overline{M}(r)$ for every $b \geqslant a$. Then $f(x)$ is improper fuzzy Riemann-integrable on $[a,\infty)$ and the improper fuzzy Riemann-integral is a fuzzy number. Further, we have:

$$\int_a^\infty f(x)dx = \left(\int_a^\infty \underline{f}(x,r)dx, \int_a^\infty \overline{f}(x,r)dx\right). \tag{5}$$

Proposition 1 [19]. If each of $f(x)$ and $g(x)$ is fuzzy value function and fuzzy Riemann integrable on $[a,\infty)$ the $f(x) \bigoplus g(x)$ is fuzzy Riemann-integrable on $[a,\infty)$. Moreover, we have

$$\int_I (f(x) \oplus g(x))dx = \int_I f(x)dx \oplus \int_I g(x)dx. \tag{6}$$

It is well-known that the H-derivative (differentiability in the sense of Hukuhara) for fuzzy mappings was initially introduced by Puri and Ralescu (1983) and it is based in the H-difference of sets, as follows.

Definition 5. Suppose $x, y \in E$. If there exists $z \in E$ such that $x = y \oplus z$, then z is called the H-difference of x and y, and it is denoted by $x -^h y$.

In this paper, the sing "$-^h$" always stands for H-difference and also note that $x -^h y \neq x \ominus y$. In this paper we consider the following definition which was introduced Bede et al. [3].

Definition 6. Suppose $f : (a,b) \to E$ and $x_0 \in (a,b)$. We say that f is strongly generalized differential at x_0 (Bede et al. [4]) if there exists an element $f'(x_0) \in E$, such that:

a) for all $h > 0$ sufficiently small, $\exists f(x_0+h) -^h f(x_0)$, $\exists f(x_0) -^h f(x_0-h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f(x_0+h) -^h f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0) -^h f(x_0-h)}{h} = f'(x_0)$$

or

b) for all $h > 0$ sufficiently small, $\exists f(x_0) -^h f(x_0+h)$, $\exists f(x_0-h) -^h f(x_0)$ and the limits

(in the metric D)

$$\lim_{h\to 0}\frac{f(x_0) -^h f(x_0+h)}{-h} = \lim_{h\to 0}\frac{f(x_0-h) -^h f(x_0)}{-h} = f'(x_0)$$

or

c)for all $h > 0$ sufficiently small, $\exists f(x_0+h) -^h f(x_0)$, $\exists f(x_0-h) -^h f(x_0)$ and the limits (in the metric D)

$$\lim_{h\to 0}\frac{f(x_0+h) -^h f(x_0)}{h} = \lim_{h\to 0}\frac{f(x_0-h) -^h f(x_0)}{-h} = f'(x_0)$$

or

d))for all $h > 0$ sufficiently small, $\exists f(x_0) -^h f(x_0+h)$, $\exists f(x_0) -^h f(x_0-h)$ and the limits (in the metric D)

$$\lim_{h\to 0}\frac{f(x_0) -^h f(x_0+h)}{-h} = \lim_{h\to 0}\frac{f(x_0) -^h f(x_0-h)}{h} = f'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively)

3. Fuzzy Laplace transform

Definition 6. Let $f(x)$ be continuous fuzzy-value function. Suppose that $f(x)\odot e^{-px}$ is improper fuzzy Riemann integrable on $[0,\infty)$, then $\int_0^\infty f(x)\odot e^{-px}dx$ is called fuzzy Laplace transforms and is denoted as:

$$L[f(x)] = \int_0^\infty f(x)\odot e^{-px}dx \ (p > 0 \ and \ integer).$$

From Theorem 1, we have:

$$\int_0^\infty f(x)\odot e^{-px}dx = \left(\int_0^\infty \underline{f}(x)\odot e^{-px}dx, \int_0^\infty \overline{f}(x)\odot e^{-px}dx)\right),$$

also by using the definition of classical Laplace transform:

$$l[\underline{f}(x,r)] = \int_0^\infty \underline{f}(x,r)e^{-px}dx,$$

and

$$l[\overline{f}(x,r)] = \int_0^\infty \overline{f}(x,r)e^{-px}dx,$$

then, we follow:

$$L[f(x)] = (l[\underline{f}(x,r), l[\overline{f}(x,r)]).$$

Theorem 2 [16]. Let $f'(x)$ be an integrable fuzzy-valued function and $f(x)$ is the primitive of $f'(x)$ on $[0, \infty)$. Then

$$L[f'(x)] = p \odot L[f(x)] -^h f(0) ,$$

where f is (a)-differentiable or

$$L[f'(x)] = (-f(0) -^h (-p \odot L[f(x)])) ,$$

where f is (b)-differentiable.

Theorem 3 [16]. Let $f(x)$ and $g(x)$ be continuous fuzzy-valued functions suppose that c_1, c_2 are constant, then

$$L[(c_1 \odot f(x)) \oplus (c_2 \odot g(x))] = (c_1 \odot L[f(x)]) \oplus (c_2 \odot L[g(x)]).$$

Theorem 4 [16]. Let f is continuous fuzzy-value function and $L[f(x)] = F(p)$, then $L[e^{ax} \odot f(x)] = F(p - a)$, where e^{ax} is real value function and $p - a > 0$.

4. Abel fuzzy integral equations

Some problems of mathematical physics are describe in terms of integral equations of the first kind. An important example is the Abel integral equation [1, 2]

$$f(x) = \int_a^x \frac{u(t)}{\sqrt{x-t}} dt, \; a \leq x \leq b, \tag{7}$$

where the kernel $k(x,t) = \frac{1}{\sqrt{x-t}}$ is singular in that $k(x,t) \to \infty$ as $t \to x$.

By Eq. (7), we see that Abel integral equations are weakly singular Volteraa integral equations of the first kind.

Abel generalized his original problem by introducing the singular integral equation

$$f(x) = \int_a^x \frac{u(t)}{(x-t)^\alpha} dt, \; 0 < \alpha < 1, \tag{8}$$

know as the Generalized Abel integral equation where α are know constants such that $0 < \alpha < 1$, $f(x)$ is a predetermined data function and $u(x)$ is the solution that will be determined. The Abel's problem discussed above is a special case of the generalized equation where $\alpha = \frac{1}{2}$. The expression $(x-t)^{-\alpha}$ is called the kernel of Abel integral equation, or simply Abel kernel. If $f(x)$ is a crisp function then the solutions of Eq. (7) are crisp as well.

However, if $f(x)$ is a fuzzy function these equations may only possess fuzzy solutions. The fuzzy integral equation which is discussed in this paper is the Abel fuzzy integral equation. Now, we introduce the parametric forms of $f(x)$, then the parametric form of fuzzy Able integral equation is as follows:

$$(\underline{f}(x,r),\overline{f}(x,r)) = \left(\int_0^x \frac{\underline{u}(t,r)}{\sqrt{(x-t)}}dt, \int_0^x \frac{\overline{u}(t,r)}{\sqrt{(x-t)}}dt\right), \tag{9}$$

for each $0 \leq r \leq 1$.

We introduce general Abel fuzzy integral equation as follows

$$(\underline{f}(x,r),\overline{f}(x,r)) = \left(\int_0^x \frac{\underline{u}(t,r)}{(x-t)^\alpha}dt, \int_0^x \frac{\overline{u}(t,r)}{(x-t)^\alpha}dt\right), \tag{10}$$

know as the generalized Abel fuzzy integral equation where α are know constants such that $0 < \alpha < 1$, $f(x) = (\underline{f}(x,r),\overline{f}(x,r))$ is a predetermined data function and $u(x) = (\underline{u}(x,r),\overline{u}(x,r))$ is the solution that will be determined.

The standard form of the nonlinear Abel fuzzy integral equation is given by

$$(\underline{f}(x,r),\overline{f}(x,r)) = \left(\int_0^x \frac{F(\underline{u}(t,r))}{\sqrt{x-t}}dt, \int_0^x \frac{F(\overline{u}(t,r))}{\sqrt{x-t}}dt\right) , \tag{11}$$

where the function $(\underline{f}(x,r),\overline{f}(x,r))$ is a given real-valued function, and $(F(\underline{u}(x,r)),F(\overline{u}(x,r)))$ is a nonlinear function of $(\underline{u}(x,r),\overline{u}(x,r))$. Recall that the unknown function $(\underline{u}(x,r),\overline{u}(x,r))$ occurs only inside the integral sign for the Abel fuzzy integral equation (11).

5. The Laplace transform method for solving Abel fuzzy integral equations

The Abel fuzzy integral equation is form Eq. (9). Then if we taking Laplace transform of both sides of Eq. (9). leads to

$$L[\underline{f}(x,r),\overline{f}(x,r)] = L[(\underline{u}(t,r),\overline{u}(t,r))]L(x^{(\frac{-1}{2})})\ for\ all\ 0 \leq r \leq 1 , \tag{12}$$

or equivalently

$$(\underline{F}(s,r),\overline{F}(s,r)) = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}}(\underline{U}(s,r),\overline{U}(s,r)) = \frac{\sqrt{\pi}}{s^{\frac{1}{2}}}(\underline{U}(s,r),\overline{U}(s,r)) , \tag{13}$$

that gives

$$(\underline{U}(s,r),\overline{U}(s,r)) = \frac{s^{\frac{1}{2}}}{\sqrt{\pi}}(\underline{F}(s,r),\overline{F}(s,r)) , \tag{14}$$

where Γ is the gamma function, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. The last equation (14) can be rewritten as:

$$(\underline{U}(s,r),\overline{U}(s,r)) = (\frac{s}{\pi}(\sqrt{\pi}s^{-\frac{1}{2}}\underline{F}(s,r)), \frac{s}{\pi}(\sqrt{\pi}s^{-\frac{1}{2}}\overline{F}(s,r))) , \tag{15}$$

which can be rewritten by:

$$L[(\underline{U}(x,r),\overline{U}(x,r))] = \frac{s}{\pi}L[(\underline{w}(x,r),\overline{w}(x,r))] , \tag{16}$$

where

$$(17)\qquad (\underline{w}(x,r),\overline{w}(x,r)) = (\int_0^x (x-t)^{-\frac{1}{2}}\underline{f}(t,r)dt, \int_0^x (x-t)^{-\frac{1}{2}}\overline{f}(t,r)dt) \ .$$

Using the fact

$$(18)\qquad L[(\underline{w'}(x,r),\overline{w'}(x,r))] = (sL[\underline{w}(x,r)-\underline{w}(0,r)], sL[\overline{w}(x,r)-\overline{w}(0,r)])$$

into (16) we obtain

$$(19)\qquad L[(\underline{u}(x,r),\overline{u}(x,r))] = \frac{1}{\pi}L[(\underline{w'}(x,r),\overline{w'}(x,r))] \ .$$

Applying L^{-1} to both sides of (19) gives the formula

$$(20)\qquad (\underline{u}(x,r),\overline{u}(x,r)) = (\frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\underline{f}(t,r)}{\sqrt{x-t}}dt, \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\overline{f}(t,r)}{\sqrt{x-t}}dt) \ ,$$

that will used for the determination of the solution $(\underline{u}(x,r),\overline{u}(x,r))$.
To determine a formula that will be used for solving the generalized Abel fuzzy integral equation (10), we will apply the Laplace transform method in a parallel manner to the approach followed before. Taking Laplace transforms of both sides of Eq. (10), leads to:

$$(21)\qquad L[(\underline{f}(x,r),\overline{f}(x,r)] = L[(\underline{u}(x,r),\overline{u}(x,r))]L[x^{-\alpha}] \ ,$$

or equivalently

$$(22)\qquad (\underline{F}(s,r),\overline{F}(s,r)) = (\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}\underline{U}(s,r), \frac{\Gamma(1-\alpha)}{s^{1-\alpha}}\overline{U}(s,r)) \ ,$$

that gives

$$(23)\qquad (\underline{U}(s,r),\overline{U}(s,r)) = (\frac{s^{1-\alpha}}{\Gamma(1-\alpha)}\underline{F}(s,r), \frac{s^{1-\alpha}}{\Gamma(1-\alpha)}\overline{F}(s,r)) \ .$$

The Eq. (23) can be rewritten as:

$$(24)\qquad L[(\underline{u}(x,r),\overline{u}(x,r))] = \frac{s}{\Gamma(\alpha)\Gamma(1-\alpha)}L[(\underline{w}(x,r),\overline{w}(x,r))] \ ,$$

where

$$(25)\qquad (\underline{w}(x,r),\overline{w}(x,r)) = (\int_0^x \frac{1}{(x-t)^{\alpha-1}}\underline{f}(t,r)dt, \int_0^x \frac{1}{(x-t)^{\alpha-1}}\overline{f}(t,r)dt) \ .$$

Using Eq. (18) and

$$(26)\qquad \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)} \ ,$$

into (24) we obtain

$$(27)\qquad L[(\underline{u}(x,r),\overline{u}(x,r))] = \frac{\sin(\alpha\pi)}{\pi}L[(\underline{w'}(x,r),\overline{w'}(x,r))] \ .$$

Applying L^{-1} to both sides of (27) gives the formula

$$(28)\qquad (\underline{u}(x,r),\overline{u}(x,r)) = \left(\frac{\sin(\alpha\pi)}{\pi}\frac{d}{dx}\int_0^x \frac{\underline{f}(t,r)}{(x-t)^{1-\alpha}}dt\ ,\ \frac{\sin(\alpha\pi)}{\pi}\frac{d}{dx}\int_0^x \frac{\overline{f}(t,r)}{(x-t)^{1-\alpha}}dt\right)\ .$$

Integrating the integral at the right side of Eq. (28) and differentiating the result we obtain the more suitable formula

(29)

$$(\underline{u}(x,r),\overline{u}(x,r)) = \left(\frac{\sin(\alpha\pi)}{\pi}\left(\frac{\underline{f}(0,r)}{x^{1-\alpha}}+\int_0^x \frac{\underline{f}'(t,r)}{(x-t)^{1-\alpha}}dt\right),\frac{\sin(\alpha\pi)}{\pi}\left(\frac{\overline{f}(0,r)}{x^{1-\alpha}}+\int_0^x \frac{\overline{f}'(t,r)}{(x-t)^{1-\alpha}}dt\right)\right)\ ,$$

for all $0<\alpha<1$.

To determine a solution for the nonlinear Abel fuzzy integral equation (11), we first convert it to a linear Abel fuzzy integral equation of the form:

$$(30)\qquad (\underline{f}(x,r),\overline{f}(x,r)) = \left(\int_0^x \frac{\underline{v}(t,r)}{\sqrt{x-t}}dt,\int_0^x \frac{\overline{v}(t,r)}{\sqrt{x-t}}dt\right)\ ,$$

by using the transformation

$$(31)\qquad (\underline{v}(x,r),\overline{v}(x,r)) = (F(\underline{u}(x,r)),F(\overline{u}(x,r))\ ,$$

where $(F(\underline{u}(x,r)),F(\overline{u}(x,r))$ is invertible, i.e $(F^{-1}(\underline{u}(x,r)),F^{-1}(\overline{u}(x,r))$ exists. This in turn means that

$$(32)\qquad (\underline{u}(x,r)),\overline{u}(x,r)) = (F^{-1}(\underline{v}(x,r)),F^{-1}(\overline{v}(x,r))\ .$$

Taking Laplace transforms of both sides of (30) leads to

$$L[(\underline{f}(x,r),\overline{f}(x,r))] = L[(\underline{v}(x,r),\overline{v}(x,r))]L[x^{-\frac{1}{2}}]\ ,$$

or equivalently

$$(33)\qquad (\underline{F}(s,r),\overline{F}(s,r)) = (\underline{V}(s,r)\frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}},\overline{V}(s,r)\frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}}) = (\underline{V}(s,r)\frac{\sqrt{\pi}}{s^{\frac{1}{2}}},\underline{V}(s,r)\frac{\sqrt{\pi}}{s^{\frac{1}{2}}})\ ,$$

that gives

$$(34)\qquad (\underline{V}(s,r),\overline{V}(s,r)) = (\frac{s^{\frac{1}{2}}}{\sqrt{\pi}}\underline{F}(s,r),\frac{s^{\frac{1}{2}}}{\sqrt{\pi}}\overline{F}(s,r))\ ,$$

The last equation (34) can be rewritten as

$$(35)\qquad (\underline{V}(s,r),\overline{V}(s,r)) = \left(\frac{s}{\pi}(\sqrt{\pi}s^{-\frac{1}{2}}\underline{F}(s,r)),\frac{s}{\pi}\sqrt{\pi}(s^{-\frac{1}{2}}\overline{F}(s,r))\right)\ ,$$

which can be rewritten by

$$(36)\qquad L[(\underline{v}(x,r),\overline{v}(x,r))] = \frac{s}{\pi}L[(\underline{w}(x,r),\overline{w}(x,r))]\ ,$$

where

$$(37)\qquad (\underline{w}(x,r),\overline{w}(x,r)) = (\int_0^x (x-t)^{-\frac{1}{2}}\underline{f}(t,r)dt, \int_0^x (x-t)^{-\frac{1}{2}}\overline{f}(t,r)dt) \ .$$

Using the fact

$$(38)\qquad L[(\underline{w'}(x,r),\overline{w'}(x,r))] = (sL[\underline{w}(x,r)-\underline{w}(0,r)], sL[\overline{w}(x,r)-\overline{w}(0,r)])$$

into (36) we obtain

$$(39)\qquad L[(\underline{v}(x,r),\overline{v}(x,r))] = \frac{1}{\pi}L[(\underline{w'}(x,r),\overline{w'}(x,r))] \ .$$

Applying L^{-1} to both sides of (39) gives the formula

$$(40)\qquad (\underline{v}(x,r),\overline{v}(x,r)) = (\frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\underline{f}(t,r)}{\sqrt{x-t}}dt, \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\overline{f}(t,r)}{\sqrt{x-t}}dt) \ ,$$

that will be used for the determination of the solution $(\underline{v}(x,r),\overline{v}(x,r))$. Having determined $(\underline{v}(x,r),\overline{v}(x,r))$, then the solution $(\underline{u}(x,r),\overline{u}(x,r))$ of (11) follows immediately by using

$$(41)\qquad (\underline{u}(x,r),\overline{u}(x,r)) = (F^{-1}(\underline{u}(x,r), F^{-1}(\overline{u}(x,r)) \ .$$

Notice that the formulas (20, 28, 29, 40) will be used for solving Abel fuzzy integral equation, and this are not necessary to use fuzzy Laplace transform method for each problem. Abel fuzzy problem given by (9, 10, 11) can be solved directly by using the formulas (20, 28, 29, 40) where the unknown function $(\underline{u}(x,r),\overline{u}(x,r))$ has been replaced by the given function $(\underline{f}(x,r),\overline{f}(x,r))$.

6. Example

Here, we consider three examples to illustrate the Fuzzy Laplace transforms methods for solving Abel fuzzy integral equations.

Example 1. Consider the following Abel fuzzy integral equation

$$(\frac{4}{3}rx^{\frac{3}{2}}, \frac{4}{3}(2-r)x^{\frac{3}{2}}) = \int_0^x \frac{u(t,r)}{\sqrt{x-t}}dt \ .$$

The exact solution in this case is given by

$$(\underline{u}(x,r),\overline{u}(x,r)) = (rx, (2-r)x) \ and \ 0 \le r \le 1 \ .$$

Notice that $\alpha = \frac{1}{2}$ and $(\underline{f}(x,r),\overline{f}(x,r)) = (\frac{4}{3}rx^{\frac{3}{2}}, \frac{4}{3}(2-r)x^{\frac{3}{2}})$, by using Eq. (20), gives:

$$(\underline{u}(x,r),\overline{u}(x,r)) = (\frac{1}{\pi}\frac{d}{dx}\left(\int_0^x \frac{\frac{4}{3}rt^{\frac{3}{2}}}{\sqrt{x-t}}dt\right), \frac{1}{\pi}\frac{d}{dx}\left(\int_0^x \frac{\frac{4}{3}(2-r)t^{\frac{3}{2}}}{\sqrt{x-t}}dt\right)) \ ,$$

then

$$(\underline{u}(x,r),\overline{u}(x,r)) = (\frac{4r}{3\pi}\frac{d}{dx}\left(\int_0^x \frac{t^{\frac{3}{2}}}{\sqrt{x-t}}dt\right), \frac{4(2-r)}{3\pi}\frac{d}{dx}\left(\int_0^x \frac{t^{\frac{3}{2}}}{\sqrt{x-t}}dt\right)) ,$$

using the fact

$$\int_0^x \frac{t^{\frac{3}{2}}}{\sqrt{x-t}}dt = \frac{3}{8}\pi x^2 ,$$

we have

$$(\underline{u}(x,r),\overline{u}(x,r)) = (\frac{4r}{3\pi}\frac{d}{dx}\left(\frac{3}{8}\pi x^2\right), \frac{4(2-r)}{3\pi}\frac{d}{dx}\left(\frac{3}{8}\pi x^2\right)) ,$$

then gives:

$$(\underline{u}(x,r),\overline{u}(x,r)) = (rx,(2-r)x) .$$

Example 2. Consider the following Abel fuzzy integral equation

$$((r^2+2r)x,(6-3r^3)x) = \int_0^x \frac{u(t,r)}{(x-t)^{\frac{2}{3}}}dt ,$$

The exact solution in this case is given by

$$(\underline{u}(x,r),\overline{u}(x,r)) = (\frac{3\sqrt{3}(r^2+2r)}{4\pi}x^{\frac{2}{3}}, \frac{3\sqrt{3}(6-3r^3)}{4\pi}x^{\frac{2}{3}}) \ and\ 0 \le r \le 1 .$$

Notice that $\alpha = \frac{2}{3}$ and $(\underline{f}(x,r),\overline{f}(x,r)) = ((r^5+2r)x,(6-r^3)x)$, by using Eq. (28), gives:

$$(\underline{u}(x,r),\overline{u}(x,r)) = (\frac{\sin(\frac{2}{3}\pi)}{\pi}\frac{d}{dx}\left(\int_0^x \frac{(r^2+2r)t}{(x-t)^{\frac{1}{3}}}dt\right), \frac{\sin(\frac{2}{3}\pi)}{\pi}\frac{d}{dx}\left(\int_0^x \frac{(6-3r^3)t}{(x-t)^{\frac{1}{3}}}dt\right)) ,$$

then

$$(\underline{u}(x,r),\overline{u}(x,r)) = (\frac{\sqrt{3}(r^2+2r)}{2\pi}\frac{d}{dx}\left(\int_0^x \frac{t}{(x-t)^{\frac{1}{3}}}dt\right), \frac{\sqrt{3}(6-3r^3)}{2\pi}\frac{d}{dx}\left(\int_0^x \frac{t}{(x-t)^{\frac{1}{3}}}dt\right)) ,$$

using the fact

$$\int_0^x \frac{t}{(x-t)^{\frac{1}{3}}}dt = \frac{9}{10}x^{\frac{5}{3}} ,$$

we have

$$(\underline{u}(x,r),\overline{u}(x,r)) = (\frac{\sqrt{3}(r^2+2r)}{2\pi}\frac{d}{dx}\left(\frac{9}{10}x^{\frac{5}{3}}\right), \frac{\sqrt{3}(6-3r^3)}{2\pi}\frac{d}{dx}\left(\frac{9}{10}x^{\frac{5}{3}}\right)) ,$$

then gives:

$$(\underline{u}(x,r),\overline{u}(x,r)) = (\frac{3\sqrt{3}(r^2+2r)}{4\pi}x^{\frac{2}{3}}, \frac{3\sqrt{3}(6-3r^3)}{4\pi}x^{\frac{2}{3}}) .$$

Example 3. Consider the following nonlinear Abel fuzzy integral equation

(42)
$$(\frac{3}{16}rt^{\frac{3}{2}}, \frac{3}{16}(2-r)x^{\frac{3}{2}}) = \int_0^x \frac{u^3(x,r)}{\sqrt{x-t}}dt ,$$

The exact solution in this case is given by

$$(\underline{u}(x,r),\overline{u}(x,r)) = (\frac{1}{4}\sqrt[3]{9rx}, \frac{1}{4}\sqrt[3]{9(2-r)x}) \ .$$

Assume $u^2(x,r)$ is invertible. The transformation

$$v(x,r) = u^3(x,r)\ ,\ u(x,r) = \sqrt[3]{v(x,r)}\ , \tag{43}$$

carries (42) into

$$(\frac{3}{16}rx^{\frac{3}{2}}, \frac{3}{16}(2-r)x^{\frac{3}{2}}) = \int_0^x \frac{v(t,r)}{\sqrt{x-t}}dt\ . \tag{44}$$

Substituting $(\underline{f}(x,r),\overline{f}(x,r)) = (\frac{3}{16}rx^{\frac{3}{2}}, \frac{3}{16}(2-r)x^{\frac{3}{2}})$ in (40) gives

$$(\underline{v}(x,r),\overline{v}(x,r)) = (\frac{3r}{16\pi}\frac{d}{dx}\left(\int_0^x \frac{t^{\frac{3}{2}}}{x-t}dt\right), \frac{3(2-r)}{16\pi}\frac{d}{dx}\left(\int_0^x \frac{t^{\frac{3}{2}}}{x-t}dt\right))\ ,$$

using the fact

$$\int_0^x \frac{t^{\frac{3}{2}}}{\sqrt{x-t}}dt = \frac{3}{8}\pi x^2\ ,$$

we have

$$(\underline{v}(x,r),\overline{v}(x,r)) = (\frac{3r}{16\pi}\frac{d}{dx}\left(\frac{3}{8}\pi x^2\right), \frac{3(2-r)}{16\pi}\frac{d}{dx}\left(\frac{3}{8}\pi x^2\right))\ ,$$

then gives:

$$(\underline{v}(x,r),\overline{v}(x,r)) = (\frac{9}{64}rx, \frac{9}{64}(2-r)x)\ .$$

This in turn gives the solutions

$$(\underline{u}(x,r),\overline{u}(x,r)) = (\frac{1}{4}\sqrt[3]{9rx}, \frac{1}{4}\sqrt[3]{9(2-r)x})\ .$$

obtained upon using (43).

7. Conclusion

In this paper, we considered linear and nonlinear Abel fuzzy integral equations. The original equation was converted into two crisp linear and nonlinear Abel integral equations. Then, we applied fuzzy Laplace transforms to obtain of the unique solution of Abel fuzzy integral equations. It was shown that this new technique is easy to implement and produces accurate results.

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