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EXISTENCE AND OSCILLATION OF SOLUTIONS OF NONLINEAR SECOND ORDER DELAY IMPULSIVE INITIAL-BOUNDARY VALUE PROBLEM CONTAINING "MAXIMUM"

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ABSTRACT. In this paper,the criteria for the existence of solutions of nonlinear second order delay impulsive initial-boundary value problem containing 'maximum' are determined through topological degree approach . Oscillation results for the system are also obtained together with two Examples given to illustrate the applications of the results obtained.

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1. INTRODUCTION

Impulsive differential equations (IDEs) are equations that are characterized by instantaneous changes in the variable describing them. The changes could be in form of jumps,shocks etc. and occur as small perturbations (impulses) at certain fixed or non-fixed moments during the process of evolution ([2],[5],[11]).

The solutions of IDEs possess discontinuities,and even,the impulsive moments describing them not only depend on some impulsive sets but also on the dynamics of the evolutionary process characterizing them ([5]&[11]). These characteristics made the study of IDEs more difficult when compared with those in differential equations. However,the IDEs offer adequate apparatus for investigating the behavours of several real life processes ([2],[15]).

Impulsive differential equations containing maximum(IDEM) provides rich platforms to study the impulsive real life processes which are described by unknown functions with the variables containing the "maximum" in the given set .The IDEs containing "maximum" are successfully used for mathematical simulation in various field of science and technology ([2],[9-10]&[14]).The investigation of these equations are rather difficult due to the discontinuous nature of their solution and the presence of maximum of the unknown functions in the IDEM describing them ([2],[10]&[14]) .

Impulsive differential equations containing maximum(IDEM) is useful in electrical engineering for designing a parallel simulator to regulate maxima deviation in current [2];useful in the planning,allocation and management of resources in a military set-up ([9-10]&[14-15]) and has potential application in aeronautics in the design of cameras for area photography using the synchronous flashing ([8])and subjecting the lens to various light intensities .Other areas of applications are in population dynamics,medicine and seismography ([8]&[10]).IDEM may contain delay such as the one studied in [10] and [14] or may be formulated as a measure differential equations as studied in ([9]&[14]).

Recent investigation in the literature revealed that many real life problems can be modelled using IDEs (for examples see [1],[4],[6]&[11]) and some of these investigations were in fact carried using oscillation theory ([4],[6]&[16]).

Gopalsamy and Zhang([4]) in 1989 published a paper which was devoted to oscillatory theory of impulsive systems and several monographs on the oscillatory theorems on IDEs are now available in the literature (for examples see [5]&[11]).

Oscillatory criteria for even order impulsive delay systems has been obtained by Lijun and Jinde[6].A comprehensive survey on oscillatory for linear and nonlinear IDEs with delays was made by Agarwal and Fatima[16].Some oscillation theorems for IDEM were obtained by Oyelami and Ale and applied to military and impulsive Fitzhugh-Nagumo models and non-linear control systems[10]. For more applications see the monograph by Oyelami and Ale[11].

In recent years, the theory of topological degree has proven to be a powerful and versatile tool in dealing with problems involving the existence and bifurcation of solutions of differential equations and control systems([7a],[12]&[15]).Topological degree as a basic tool, has been applied successfully in obtaining results on ordinary, functional and partial differential equations in generalized settings([7a]&[11]).New applications of Leray–Schauder theory and its extensions have also been given, specially in bifurcation theory,nonlinear boundary value problems and equations in ordered spaces([7b]).

It must be noted that topological degree as tool gives a more flexible and sophisticated technique for establishing the existence of solution of operator equations in comparison with fixed point techniques for the compact operators, which have building blocks from topological algebra.

In this paper,we will make use of the generalized Leray- Schauder topological degree theorem to obtain criteria for the existence of solutions of impulsive initial boundary value problems. We must note that this is an extension of our earlier results on IDEs([12]) to the nonlinear second order delay impulsive initial-boundary value differential equations containing 'maximum'.It is also the extension of our work in [10] to second order impulsive system containing maximum with problem under consideration will be approached from topological degree point of view as against accretive map used in [10]. We also intend to derive oscillation results for(NDIIBDEM) with two Examples given to illustrate the applications of the results obtained.

Finally, the following preliminary definitions and notations would be useful in our study:

2. Preliminary definitions and Notations

Let $R^+ = [0,\infty)$ and R^n be the n-dimensional Euclidean space with elements $x = (x_1, x_2, x_3 ... x_n)$ and equipped with the norm $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$ .Let the sequence of impulsive moments be $\{t_k\}, k = 0, 1, 2, ...$ such that $0 < t_1 < t_2 < ... < t_k, \lim_{k\to\infty} t_k = +\infty$.

Let $C(R^+, R^n)$ be the set of continuous functions defined on $R^+$and taking values in R^n.

$$PC(R^+, R^n) = \left\{ \begin{array}{c} y(t) : y(t) \in C(R^+ - \{t_k\}, R^n) \text{ and } \lim_{t\to t_k+0} y(t) \\ \text{exists and it is equal to } y(t_k) \end{array} \right\}$$

In R^+ define the set of intervals $I_n = [t_{k-1}, t_k)$ and $J_k = (t_k - h, t_{k-1}], k = 0, 1, 2, ...$ and $M_n(R^+)$ be the $n \times n$ matrix defined on R^+.

Now consider the nonlinear second order delay impulsive initial-boundary value differential equations containing 'maximum'(NDIIBDEM)

$$\left. \begin{array}{c} \ddot{x}(t) + f(t, x(t), \dot{x}(t), x(t-h)) = g(x(t), \max_{s\in I_0} x(s)), t \neq t_k, k = 0, 1, 2, ... \\ x(t_k + 0) = L_k x(t_k - 0), k = 0, 1, 2, ... \\ \dot{x}(t_k + 0) = L_k^1 x(t_k - 0), k = 0, 1, 2, ... \\ Mx_a + Nx_b = C, x_a = x(t = a), x_b = x(t = b) \\ \dot{x}(t_0 + 0) = x_1 \end{array} \right\} \quad (1)$$

Where $M, N \in M_n(R^n), C \in R^n, x(t) \in PC(R^+, R^n).\{L_k\}$ and $\{L_k^1\}$ are sequences of real numbers such that the maximum $\max_{t\in I_0} x(t)$exists in I_0.

We will also make use of the following notations and definitions:

$\lambda(.)$ is the measure of non compactness of a bounded set(.) with the property that $\lambda(F(A,B,C)) \leq L_1\lambda(A)+L_2\lambda(B)+L_3\lambda(C)$ where $L_i, i=1,2,3$ are constants A,B,C are some bounded sets([11]&[13])

Definition 1. ***: Fredholm map***

Let $X,\ Z$ be real normed spaces and denote by $|\cdot|$ the corresponding norms. A linear mapping $L: domL \subset X \longrightarrow Z$ will be called a Fredholm map if the following conditions are satisfied:

(a) If the image of L(i.e.,$im\ L$) is closed and has a finite codimension.
(b) The dimension of kernel of L(i.e., $\ker\ L$) is finite (i.e. $dim\ ker\ L < +\infty$).

Where $dim(.)$ is the dimension of (.) and $Dom(L)$ is the domain of the map $L.$

Remark 1

The definition of codimension can be understood from the following Lemma1 .We state without proof because it is available in standard texts on functional analysis:

Lemma 1 : Fredholm index

The Fredholm index of a Fredholm map will be define as

$$\begin{aligned} ind\ L &= \dim\ \ \ker L - co\dim(in\ L) \\ &= \dim\ker\ L - \dim\left(\tfrac{Z}{im\ L}\right) \\ &= dim\ \ker L - dim(co\ker nalco\ker\ L) \end{aligned}$$

Where $co\dim(.)$ is the codimension of $(.)$ and $ind\ L$ is the index of the Fredholm map

Lemma 2

A fundamental relation often obeyed by the Fredholm map is:

$$co\dim(imL) = \dim\left(\frac{Z}{im\ L}\right) = \dim(\text{Cokernal Coker L}). \qquad (2)$$

Therefore,it follows that $ind\ L = \dim(\ker L - \dim(co\ker nel)co\ker L)$.

From standard results from linear functional analysis, it can be established from the Fredholm maps, that there exist continuous projectors P and Q(see for example

,Mawhim [7a], pp. 6) such that

$$P : X \longrightarrow X,\ Q : Z \longrightarrow Z$$
$$\text{implies that } imP = kerL,\ kerQ = imL$$

Hence

$$X = kerL \oplus kerP \text{ and } Z = imL \oplus imQ$$

as the topological direct sums. That is, X is direct sum of the kernels of L and P while Z is direct sum of the images of L and Q respectively. The restriction L_p of L to $dom\ L \cap kerP$ is isomorphic to imL. i.e., $L_p \subset dom\ L \cap ker\ P$ isomorphic to imL and the algebraic inverse $K_p : im\ L \longrightarrow dom\ L \cap ker\ P$ is defined. Denote by $K_p : Z \longrightarrow domL \cap kerP$, the generalized inverse of L,and it is defined by

(3) $$K_{P,Q} = K_P(I - Q).$$

Definition 2. ***: L-compactness***

Let $L : dom\ L \subset X \longrightarrow Z$ be a Fredholm map and let E be a metric space and $G : E \longrightarrow Z$ be a map. Then G is L-compact on E if the maps $QG : E \longrightarrow Z$ and $K_{P,Q}G : E \longrightarrow X$ are compact on E.

That is, continuous on E such that $QG(E)$ and $K_{P,Q}G(E)$ are relatively compact.

Definition 3. ***:*** L***completely continuous***

$G : X \longrightarrow Z$ will be said to be Lcompletely continuous, if E is L-compact for every bounded $E \subset X$.

The condition(A) is said to be satisfied if the following conditions are satisfied:

$f, g \in C(R^+, R^n)$ and lipchitz with respect to the second and the third variable respectively while g is lipchitz with respect to all its arguments.

2.1. **Comparison equations.** Consider the following comparison equation

$$\left.\begin{array}{c} \ddot{u}(t) = g(u(t), \max_{t\in I_0} u(t)), t \neq \gamma_k, k = 0, 1, 2, ... \\ \Delta u(\gamma_k) = \beta_k u(\gamma_k) \\ \Delta u^{/}(\gamma_k) = \beta_k^{/} u(\gamma_k) \end{array}\right\} (CIDECM)$$

Where β_k and $\beta_k^{/}$ are some constants and

$g : R^+ \times W_2 \to R^n, u(t) \in C^2(R^+, R^n) \cap P(R^+, R^n)$.

Definition 4.

A function $x(t) \in C^2(R^+, R^n) \cap P(R^+, R^n)$ is to be a solution to the (DIDEM) in eq.(1) if it satisfies it along with prescribed initial and boundary conditions in the given interval:if $x(t) > 0$ for every $t \in I_0$,we say that the solution is eventually positive in the given interval and if $x(t) < 0$,then the solution is said to be eventually negative in the given interval .If $x(t) = 0$ for infinitely many $t \in R^+$ then the solution is sad to be oscillatory.

2.2. **Homotopy and functional equations.** Let $F(x, \lambda) = Lx + (1 - \lambda)H$ then F is said to be homotopy invariant if $F(., 0) = H, F(., 1) = L$ for $\lambda \in J_0$.

The eq. (1) is equivalent to the following functional equation

(4)
$$Kz = \omega$$

Where $K = (\frac{d^2}{dx^2}, \Delta, \Delta_1), \omega = (g - f, L_k, L_n^/)^T$ and $z = (x(t), x^/(t))^T$, T is the transpose of the vector.

Let $x(t)$ be a solution of eq. (1) and define $K^{-1}z(t) = z(t)$.Therefore, we deduce that

$\ker K = \{z(t) \in \text{dom}(K)\text{: } z(t) \text{ is constant map such that } L_k = L_k^/ = 0\}$
$imK = \{(z^/(t), Kz(t)) \in G, z(t) \in im(K)\}$
$= A^{*-1}\{imK\}$
Where

(5)
$$A^*w = x_0^/ + x_0 + \Pi_{t_0<t_k<}(1 + L_k^/) + \int_0^t (t - s)\Pi_{t_0<t_k<}(1 + L_k)w(s)ds$$

$w = w(t) := g(t, x(t), \max_{t \in I_0} x(t)) - f(t, x(t), x(t - h))$
And $w_0(t) = w(t)$ when $f(t, x(t), x(t - h)) \geq 0$.

Moreover, the functional equation for the comparison equation can be constructed as

$K_1z = w, K_1 = (\frac{d^2}{dx^2}, \Delta_c, \Delta_c^1), \omega = (g, \beta_k, \beta_k^1)$ and $z = (u(\gamma), u^1(\gamma))$.

3. Main Results

Theorem 1

Let the condition (A) be satisfied then Fredholm map has zero index.

Proof

To establish the proof we need to show that A^* in eq.(5) is onto Z,its kernel, Ker A^* is closed and the index of K is zero.

We proceed as follows

dim $\ker K = \dim\ker(L_k) = 0$

codim K **=**$n - \dim im(K) = \dim\ker(K)$

$= \dim\ker K$

$ind\ K = \dim\ker(K) =$**codim** $K = n - n = 0.$

Next, we show that $im\ K$ is closed. For this, let $\{y_n\} \in im\ (K)$ such that $y_n \to y$ as $n \to \infty$ .It is enough for us to show that $y \in im(K)$. Suppose on the contrary that, $y \notin im\ (K)$ then for every $\in > 0$ there exist two integers n_0 and n such that $n_0 > n$ and $|Ky_n - Ky| > \in$ such that

$\lim_{n\to\infty} |y_n - y| = 0.$

Therefore

$$|Ky_n - Ky| = |x_{no} - x^{/}_{no}| + |x_{no} - x^{/}_{no}|| \prod_{t_0<t_k<t} (1 + L^{/}_k)| + \int_0^t (t-s) \prod_{t_0<t_k<t} (1 + L_k)||w_n(s) - w(s)|ds \quad \textbf{(6)}$$

From the Condition (A) there exist constants $k_i, i = 1, 2, 3$ such that

$$|w_n(t) - w(t)| \leq k_1|y_n(t-h) - y(t-h)| + k_2(\max_{t\in I_0} y_n(t) - \max_{t\in I_0} y(t))$$

.

But by the results in [9] and [10] there exist k_4 such that

$$\max_{t\in I_0} y_n(t) - \max_{t\in I_0} y(t) < k_4 \max_{t\in I_0}(y_n(t) - y(t))$$

Now let $u_n(t) = |w_n(t) - w(t)|$ and $v_n(t) = |y_n(t) - y(t)|$

Therefore,

$$u_n(t) \leq k_1 v_n(t-h) + k_2 k_4 \max_{t\in I_0} u_n(t) + k_3 v(t)$$

As $n \to \infty$, $u_n(t) \to u(t)$ and $v_n(t) \to v(t) = 0$ and $w_n(t) \to w(t)$.

It implies that $0 < u_0(t) = \lim_{n\to\infty} v_n(t) \leq 0$,hence $u(t) = 0$.

Therefore,$|Ky_n(t) - Ky(t)| \to 0$ as $y_n(t) \to y$ as $n \to \infty$ hence $0 < \in < |Ky_n(t) - Ky(t)| \to 0$ as $n \to \infty$ this is a contradiction. Therefore,if $\{y_n(t)\} \subset$ im K such that $y_n(t) \to y(t)$

as $n \to \infty$ it implies that $y \in$im K.Therefore im K is closed and it is a Fredholm map of degree zero. Since K is a Fredholm map with zero index then by Lemma 1 there exist two projects P and Q such that $imL = \ker Q, imP = kerL$ and that $\ker Q \oplus \ker K = X$ and $imK \oplus imP = Z.$

Let

$$Tx(t) = S(x(t)) + x_0^{/} + x_0 \prod_{t_0<t_k<t} (1 + L_k^{/}) \int_0^t (t-s) \prod_{t_0<t_k<t} (1 + L_k^{/}) w_0(s) ds \tag{7}$$

And

$$P_S(x(t)) = S(x(t)) \tag{8}$$

Then by standard result on Fredholm maps

$$K_{Ps} g^* = K_S^{-1} A^* g^* \tag{9}$$

And

$$K_{P_S, Q_f}\, g = K_S^{-1} \Xi A^* g^* \tag{10}$$

Where K_{P_S} and Q_f and the generalized inverse with respect to the projectors P_S and Q_f respectively and K_S is the restriction of K to $\ker S$. We can construct K_{P_S} as

$$K_{P_S} g^* = K_S^{-1}[x_0^{/} + x_0 \prod_{t_0<t_k<t} (1 + L_k^{/}) \int_0^t (t-s) \prod_{t_0<t_k<t} (1 + L_k^{/}) w_0(s) ds] \tag{11}$$

$$K_{P_S, Q_\Xi}\, g^* = K_S^{-1} \Xi A^* g^* \tag{12}$$

Such that K_{P_S} and K_{P_S, Q_Ξ}, Q_Ξ are L – compacts.

Proof

First of all we note that A^* is continuous,thus if $B_0(r)$ and $B_1(r)$ are two open sets such that $A^* : B_0(r) \subset B_1(r)$ takes $z_0 \in B_0(r)$ to $z_1 \in B_1(r)$ such that $A^* z_0 = z_1$ then $K_S^{-1} A^* z_0 = K_S^{-1} z_1 = \omega \in B_0(r)$ and also $K_S \omega = z_1 \in B_1(r)$.

Therefore $K_{P_S} B_0(r) \subset B_1(r)$ which shows that K_{P_S} takes an open set to an opens set hence it as continuous. We claim that it is equi-continuous and equi-bounded too. We establish this claim by the use of Ascolis-Arzela's theorem as follows:

$|K_S^{-1} A^* g^*(t_p) - K_S^{-1} A^* g^*(t_{p-1})|$

$$\leq |K_S^{-1}||[\int_0^{t_p} t_p \prod_{t_0<t_k<t} (1 + L_k) w(s) ds|$$

$$- \int_0^{t_p} t_p \prod_{t_0<t_k<t} (1 + L_k) w(s) ds]| \to 0$$

As $t_{p-1} \to t_p$ for $t_p \in [t_0, t_k), k = 1, 2, \cdots$

Now, let τ = max $[|x_0^{/}| + |x_0| \prod_{t_0<t_k<t} (1 + L_k)]$

Then

$$\lambda(K_S^{-1} A^* g^*) \leq \tau + \tau^2 + |b - a| \tau \lambda(w(t))$$

Since

$$\lambda(w(t)) \leq (m_1 + m_2 + lm_3 + x_{\max}) \lambda(x(t))$$

if $\tau + \tau^2 + |b - a| \tau (m_1 + m_2 + lm_3 + x_{\max}) < 1$

Then $\lambda(K_S^{-1} A^* g^*) \leq \gamma \lambda(x(t))$ hence K_{P_S} is set contractive(bounded and continuous) (see[10]). Hence compactness follows from Ascolis – Arzela's theorem. We can also show that K_{P_S, Q_Ξ} is also compact by similar argument,hence $K_{P_S}, K_{P_S, Q_\Xi}$and Q_Ξ are L – compact.

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Theorem 2 (Existence Theorem)

Let the following conditions be satisfied:

H1:$x(t) \in C^2(R^+, R^n) \cap PC(R^+, R^n)$

H2: f and g in the eq. (1) satisfies the Condition (A)

H3: L_k and $L_k^{/}$ in the eq. (1) are such that $\sum_{k=0}^{\infty} L_k < \infty$ and $\sum_{k=0}^{\infty} L_k^{/} < \infty$.

Then there exist at least one solution to the eq.(1) existing in the interval I_k.

Proof.

From the homotopy equation $F(x, \lambda) = \lambda K x + (1 - \lambda) H x$ such that

$F(., 0) = H, F(., 1) = K, \lambda \in [0, 1]$.

We will establish the proof by the using the generalized Leray –Schauder's theorem (see [7a,b]). Thus it is enough to show that if $D_k(F, \Omega) \neq 0$ and K is a prior bounded.

By Brouwer degree theorem, we that

$D_K(F, \Omega_r) = D_K(F(0, ., \Omega_r) = D_K(F(1, .), \Omega_r) = D_K(F, \Omega_r) \neq 0.$

Since H is K-compact map and F is homotopy invariant for an open set $\Omega_r \subset R$. To complete the proof, it suffix to show that F is a priori bounded. Suppose on contrary that it is not a priori bounded then there exist$\{(\lambda_n, x^n)\}$in $C[0, 1] \times PC(R^+, R^n)$ such that for $|F(\lambda^n, x^n)| > \rho$ for $|x^n| < \in, |\lambda^n| < \in$ for every $\in > 0$ and $n > N, n$ is an integer. Let $\lambda = \lim_{n\to\infty} \lambda^n$ be finite since H and K are compact operators thus $|H| = \lim_{n\to\infty} \sup_{|x^n|=1} \frac{|Hx^n|}{|x^n|} < \infty$ and $|K| = \lim_{n\to\infty} \sup_{|x^n|=1} \frac{|Kx^n|}{|x^n|} < \infty$.For every $\in > 0$ and $n > N, N \in \mathbb{Z}$, pick $\lambda = \frac{\rho}{|K|-|H|\in}$ thus $\rho < \lambda^n|Kx^n| + (1-\lambda^n)|Hx^n|$,hence $n \to \infty$ we have $\rho < \lambda|Kx| + (1-\lambda)|Hx| < \rho$ a contradiction. Therefore $F(\lambda, x)$ is a priori bounded and has a fixed point by Leray Schauder's fixed point theorem. Thus $Kx(t) = x(t) = x$ and by Lemma 2 , $x(t)$ is the solution to eq.(1).

This ends the proof.

Theorem 3 (uniqueness theorem)

Let the following conditions be satisfied :

H_1 :There exist constants $k_1$and k_2 such that

$$|g(t, x(t), \max{}_{t\in I_0} x(t)) - g(t, y(t), \max{}_{t\in I_0} y(t))| \\ \leq k_1|x(t) - y(t)| + k_2|\max_{t\in I_0} x(t) - \max_{t\in I_0} y(t)|$$

$x(t), y(t) \in PC(R^+, R^n)$

H_2 : There exist constants k_3 and k_4 such that

$$|f(t, x(t), x(t-h)) - f(t, y(t), y(t-h))| \\ \leq (k_3 + k_4e^{-h})|x(t) - y(t)|$$

For $x(t), y(t) \in PC(R^+, R^n), h \simeq 1 + \frac{k_0k_1+k_3}{k_4}$.

H_3 : Let N be a positive constant such that

$N = \max[1, |\prod(1 + L_k^/)|, |\prod(1 + L_k)|]$

Then the solution of eq.(1) is uniquely determined in I.

Proof

Let $x(t)$ and $y(t)$ be solution of eq.(1) satisfied the initial condition $x(0) = x_0, y(t) = y_0, x^/(0) = x_0^/$ and $y^/(t) = y_0^/$.Moreover,let $z(t) = x(t) - y(t)$ and assume that $y(t-h) = e^{-h}y(t)$.

Then

$$\begin{aligned} z(t) &= z_0^/ + z_0\Pi_{t_0<t_k<}(1 + L_k^/) \\ &+ \int_0^t (t-s)\Pi_{t_0<t_k<}(1 + L_k)g(t, x(t), \max{}_{t\in I_0} x(t)) - g(t, y(t), \max{}_{t\in I_0} y(t))ds \\ &+ \int_0^t (t-s)\Pi_{t_0<t_k<}(1 + L_k^/)f(t, x(t), x(t-h)) - f(t, y(t), y(t-h))ds \end{aligned} \tag{13}$$

Therefore

$$|z(t)| \leq M + k_1N\int_0^t (t-s)(\max_{t\in I_0} x(s) - \max_{t\in I_0} y(s))ds$$

$M := \max[|z_0|, |z_0^/|, |\prod_k(1 + L_k^/)|, |\prod_k(1 + L_k)|]$

Also note that (see[9]&[10]) $\max_{t\in I_0} x(t) - \max_{t\in I_0} y(t) \leq k_0(\max_{t\in I_0}(x(t) - y(t)))$

Then we can show that $|z(t)| \leq M + N(k_0k_1 + k_3 + k_4e^{-h})e^{\frac{3t^2}{2}}$.

Let $e^{-h} \simeq 1 - h$ and from M and the condition imposed on h we have

$|x(t) - y(t)| \leq (|x_0^{/} - y_0^{/}| + |x_0 - y_0|| \prod_k (1 + L_k^{/})|e^{\frac{3}{2}t^2}$.

Then uniqueness follows since $x_0 = y_0$ and $x_0^{/} = y_0^{/}$.

f and g are assumed to satisfied lipschitz conditions and the delay must be very small and also satisfy the condition $h \simeq 1 + \frac{k_0k_1+k_3}{k_4}$ for the solution to be unique.

Corollary 1(Existence of solution to the comparison equations)

Let the following conditions be satisfied:

H1:$u(t) \in C^2(R^+, R^n) \cap PC(R^+, R^n)$

H2: g in the (CIDECM) satisfies the Condition (A) for $f = 0$

H3: β_k and β_k^1 in the (CIDECM) are such that $\sum_{k=0}^{\infty}[\beta_k + \beta_k^1] < \infty$.

Then there exist at least one solution to the(CIDECM) existing in the interval I_k.

Proof

Straight forward like Theorem 1.

3.1. **Oscillation Theorems.** If $w(t) := g(t, x(t), \max_{t\in I_0} x(t)) - f(t, x(t,)x(t-h)), w(t) = w_0(t)$when $f(t, x(t), x(t-h)) \geq 0$

Therefore eq.(1) becomes

$$\left.\begin{array}{c} \ddot{x}(t) = w(t), t \neq t_k, k = 0, 1, 2, ... \\ \Delta x(t_k) = \beta_k x(t_k) \\ \Delta x^{/}(t_k) = \beta_k^{/} x(t_k) \end{array}\right\} (14)$$

$0 < t_1 < t_2 < ... < t_k, \lim_{k\to\infty} t_k = +\infty$.

Integrate eq.(14) we get

$$x(t) = x_0 + x_0^{/} \prod_k (1 + L_k^{/}) + \int_0^t (t - s) \prod_{t_0<t_k<t} (1 + L_k^{/}) w(s) ds \quad (15)$$

and

$$\begin{aligned} Mx_a + Nx_b &= (N-M)[x_0 + x_0^{/}\prod_k (1+L_k^{/}) \\ &+ \int_0^a (a-s) \prod_{t_0<t_k<t} (1+L_k^{/}) Mw(s)ds \\ &+ \int_0^b (b-s) \prod_{t_0<t_k<t} (1+L_k^{/}) Nw(s)ds \end{aligned} \tag{16}$$

Theorem 4 (Oscillation Theorem)

Suppose that the following conditions are satisfied:

H_1 :(i) $f(t, -x(t), x(t-h)) = -f(t, x(t), x(t-h))$

(ii) $f(t, x(t), x(t-h)) \geq 0$ for $x(t) \geq 0, x(t-h) \geq 0$

(iii) $L_k \geq 0, L_k^{/} \geq 0, \sum_{k=1}^{\infty} L_k < \infty$ and $\sum_{k=1}^{\infty} L_k^{/} < \infty$

H_2 : $\lim_{t\to\infty} \inf \frac{1}{t} \int_0^t (1-\frac{s}{t}) \prod_{t_0<t_k<t} (1+L_k)|w_0(s)|ds = -\infty$

H_3 :$\lim_{t\to\infty} \sup \frac{1}{t} \int_0^t (1-\frac{s}{t}) \prod_{t_0<t_k<t} (1+L_k)|w_0(s)|ds = \infty$

Then the solution of IDECM is oscillatory in I_k.

Proof:

Suppose $x(t)$ is the solution of eq.(1) passing through x_0 in the interval I_k and if such that $x(t) > 0$ for $t \in I_k$ then by hypotheses H_1(ii)

$\frac{x(t)}{t} \leq \frac{x_o}{t} + \frac{x_o^{/}}{t} \prod_{t_0<t_k<t} (1+L_k) + \int_0^t (1-\frac{s}{t}) \prod_{t_0<t_k<t} (1+L_k^{/}) w_0(s)ds$

Thus as $t \to \infty$ the first and the second terms in above inequality tends to zero while the last term is unbounded below, a contradiction of $x(t) > 0$, $t > t_0$, $t \in I$. Take $x(t) < 0$,we have

$$-\frac{x(t)}{t} < \frac{-x_0}{t} - \frac{x_0^{/}}{t} \prod_{t_0<t_k<t} (1+L_k) - \int_0^t (1-\frac{s}{t}) \prod_{t_0<t_k<t} (1+L_k^{/}) w_0(s)ds$$

As $n \to \infty$ the first two terms tend to zero while the last term is unbounded above,a contradiction of $x(t) < 0, t > t_0, t \in I$.Hence $x(t) = 0$ for infinitely many $t \in I$.Hence the proof. Δ

We investigate condition for oscillation of eq.(1).

Let $\alpha = x_0^{'} + x_0 \Pi(1 + L_k^{/})$,

$A^t = \int_0^t (t-s)\Pi(1+L_k^/)w_0(s)ds,$

And

$B^t = \int_0^t (t-s)\Pi(1+L_k^/)f(t,x(t),x(t-h))ds$

Therefore, from eq.(1),for $a_i, b_i \in I, i = 0,1,2,...$if $x(t)$ is the solution of eq.(1) passing through $a_i, b_i \in I$.

Then $x(a_i) = \alpha + A^{a_i} + B^{a_i}$ and $x(b_i) = \alpha + A^{b_i} + B^{b_i}$.Hence $x(a_i)x(b_i) = \alpha^2 + C\alpha + D$ where $C = A^{a_i} + B^{a_i} + A^{b_i} + B^{b_i}$ and $D = (A^{a_i} + B^{a_i})(A^{b_i} + B^{b_i})$.For oscillation to happen there must be $t \in [a_i, b_i] \subset I$ such that $x(a_i)x(b_i) < 0$ for infinite many i.

This means solving the inequality

$$\alpha^2 + C\alpha + D < 0$$

which has the solution such that

$$\frac{-C-\sqrt{C^2-4D}}{2} < \alpha < \frac{-C+\sqrt{C^2-4D}}{2}$$

Since $\sqrt{C^2-4D} \approx C(1-\frac{2D}{C^2}) = C - \frac{2D}{C}$.

After some manipulations,we obtain that the system will be oscillatory if

$$\frac{D}{C} < -[x_0^{'} + x_0\Pi(1+L_k^/)] < \frac{C^2+2DC}{2C}$$

4. Examples

Example 1

Investigate the oscillatory property of the following impulsive system:

$$\left.\begin{aligned} \ddot{x}(t) + e^t \sin x(t) = (a\sin 2\pi t + b\cos 2\pi t)\max_{t\in I_0} x(t), \quad t \neq t_k, k = 1,2,3,... \\ \triangle x(t_k) = \frac{1}{2^k}x(t_k), \quad k = 1,2,\dots \\ \triangle x'(t_k) = \frac{1}{2^k}x'(t_k), \ k = 1,2,\dots \end{aligned}\right\} (E_1)$$

Solution

Since $e^t \sin x(t)$ is odd in $x(t)$, $L_k = L_k^{'} = \frac{1}{2^k} \geq 0$

$\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \frac{1}{1-\frac{1}{2}} = 2 < \infty$

Therefore,it is easy to show that

$\limsup_{t\to\infty} \left[\frac{1}{t}\int_0^t \left(1-\frac{s}{t}\right)\prod_{t_0<t_k<t}(1+L_k)(a\sin 2\pi t + b\cos 2\pi t + e^{2t})|\max_{s\in I_0} x(s)|ds\right] = +\infty$

And also

$\liminf_{t\to\infty} \left[\frac{1}{t}\int_0^t \left(1-\frac{s}{t}\right)\prod_{t_0<t_k<t}(1+L_k)(a\sin 2\pi t + b\cos 2\pi t + e^{2t})|\max_{s\in I_0} x(s)|ds\right] = -\infty$

Furthermore,using the comparison equation

$$\ddot{u}(t) + e^t \sin u(t) \leq a + b + e^{2t} \max_{s \in I_0} u(s), \gamma \neq \gamma_k, k = 1, 2, 3, ...$$
$$\Delta \dot{u}(\gamma_k) \leq u'(\gamma_k)$$
$$\Delta u(\gamma_k) \leq u(\gamma_k)$$

The solution is oscillatory for $t \geq t_0 = 0$.

Example 2

Consider the following impulsive system containing "maximum"

$$\left.\begin{array}{c} \ddot{x}(t) + ax(t) + d + x(t)\sum_{j=1}^{\infty} b_j x(t - t_j) = g(x(t), \max_{s \in I_0} x(s)), t \neq t_k, k = 0, 1, 2, ... \\ x(t_k + 0) = L_k x(t_k - 0), k = 0, 1, 2, ... \\ \dot{x}(t_k + 0) = L_k^1 x(t_k - 0), k = 0, 1, 2, ... \\ Mx_a + Nx_b = C, x_a = x(t = a), x_b = x(t = b) \\ \dot{x}(t_0 + 0) = x_1 \end{array}\right\} (E_2)$$

Where $M, N \in M_n(R^n), C \in R^n, x(t) \in PC(R^+, R^n).\{L_k\}$ and $\{L_k^1\}$ are sequences of real numbers such that the maximum $\max_{t \in I_0} x(t)$ exists in I_0.moreover,a, d and b_j are nonnegative real numbers.

We will investigate the existence of the solution of the above system.

Solution

Assume that $x(t - t_j) = e^{-\lambda t_j} x(t)$ such that $\sum_{j=1}^{\infty} b_j e^{-\lambda t_j} < \infty$ and let

$f(t, x(t), x(t - h)) = ax(t) + d + x(t)\sum_{j=1}^{\infty} b_j x(t - t_j)$ and let there exists a constant k_2 such as $k_2 = \max[|x(t)|, y(t)|]$ for $x(t), y(t) \in PC(R^+, R^n)$.

Therefore

$$|f(t, x(t), x(t - h)) - f(t, y(t), y(t - h))| \leq a|x(t) - y(t)| + k_2 \sum_{j=1}^{\infty} b_j e^{-\lambda t_j} |x(t) - y(t)|$$
$$= (a + k_2 \sum_{j=1}^{\infty} b_j e^{-\lambda t_j})|x(t) - y(t)|.$$

Suppose that there exist k and k_2 are constants such that

$$|g(x(t), \max_{s \in I_0} x(s)) - g(y(t), \max_{s \in I_0} y(s))| \leq k_1|x(t) - y(t)| + k_2|\max_{t \in I_0} x(t) - \max_{t \in I_0} y(t)|$$

And $k_0 = \max[1, \prod_{t_0 < t_k < t}(1 + L_k), \prod_{t_0 < t_k < t}(1 + L_k^{/})]$.

Then by Theorem 3, the solution of E_2 exists and it is uniquely determined in I if

$h = \sum_{j=1}^{\infty} \ln(\frac{b_j}{\lambda t_j}) \approx 1 + \frac{k_0 k_1 + k_2}{k_4}$.

We investigate the oscillatory behaviour of the system as follows:

Let $f(t, x(t), x(t-h)) = (a + \sum_{j=1}^{\infty} b_j x(t-t_j))x(t) + d$ such that $d \geq 0$ and $\sum_{j=1}^{\infty} b_j x(t-t_j))x(t) + d \geq 0$. By Theorem 4,the solution will be oscillatory in I

if $f(t, -x(t), x(t-h)) = -f(t, x(t), x(t-h))$, that is, if and only if $x(t) = 0$ or $a = 0$ and couple with the fact that the following conditions are satisfied:

C_1 : $\lim_{t\to\infty} \inf \frac{1}{t} \int_0^t (1 - \frac{s}{t}) \prod_{t_0 < t_k < t} (1 + L_k)|w_0(s)|ds = -\infty$

C_2 : $\lim_{t\to\infty} \sup \frac{1}{t} \int_0^t (1 - \frac{s}{t}) \prod_{t_0 < t_k < t} (1 + L_k)|w_0(s)|ds = \infty$.

This can be possible if $g(x(t), \max_{s\in I_0} x(s))$ can be constructed in such a way that the conditions in C_1 and C_2 above are satisfied.

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